OF MINNESOTA TWIN CITIES

Linear algebra methods for data mining

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Introduction: a few factoids

- Data is growing exponentially at an "alarming" rate:
 - 90% of data in world today was created in last two years
 - Every day, 2.3 Million terabytes (2.3 $\times 10^{18}$ bytes) created
- Mixed blessing: Opportunities & big challenges.
- Trend is re-shaping & energizing many research areas ...
- including my own: numerical linear algebra

Introduction: What is data mining?

Set of methods and tools to extract meaningful information or patterns from data. Broad area : data analysis, machine learning, pattern recognition, information retrieval, ...

Tools used: linear algebra; Statistics; Graph theory; Approximation theory; Optimization; ...

This talk: brief introduction – with emphasis on linear algebra viewpoint

+ our initial work on materials.

Emphasis on "Dimension reduction methods"

Drowning in data



Picture modified from http://www.123rf.com/photo_7238007_man-drowning-reaching-out-for-help.html

Major tool of Data Mining: Dimension reduction

- Goal is not as much to reduce size (& cost) but to:
- Reduce noise and redundancy in data before performing a task [e.g., classification as in digit/face recognition]
- Discover important 'features' or 'paramaters'

The problem:Given: $X = [x_1, \cdots, x_n] \in \mathbb{R}^{m \times n}$, find aIow-dimens. representation $Y = [y_1, \cdots, y_n] \in \mathbb{R}^{d \times n}$ of XAchieved by a mapping $\Phi : x \in \mathbb{R}^m \longrightarrow y \in \mathbb{R}^d$ so: $\phi(x_i) = y_i, \quad i = 1, \cdots, n$



- \blacktriangleright may be linear : $y_i = W^{ op} x_i$, i.e., $Y = W^{ op} X$, ...
- ... or nonlinear (implicit).
- > Mapping Φ required to: Preserve proximity? Maximize variance? Preserve a certain graph?

Example: Principal Component Analysis (PCA)

In *Principal Component Analysis* W is computed to maximize variance of projected data:

$$egin{aligned} & \max & & \sum \limits_{i=1}^d \left\|y_i - rac{1}{n} \sum \limits_{j=1}^n y_j
ight\|_2^2, \ y_i = W^ op x_i. \ & W^ op W^ op W = I \end{aligned}$$

Leads to maximizing

$${
m Tr}\left[W^ op(X-\mu e^ op)^ op W
ight], \hspace{1em} \mu=rac{1}{n}\Sigma_{i=1}^n x_i$$

> Solution $W = \{ \text{ dominant eigenvectors } \}$ of the covariance matrix == Set of left singular vectors of $\bar{X} = X - \mu e^{\top}$

SVD:

$$ar{X} = U \Sigma V^ op, \quad U^ op U = I, \quad V^ op V = I, \quad \Sigma = \mathsf{Diag}$$

> Optimal $W = U_d \equiv$ matrix of first d columns of U

> Solution W also minimizes 'reconstruction error' ..

$$\sum_i \|x_i - WW^T x_i\|^2 = \sum_i \|x_i - Wy_i\|^2$$

ln some methods recentering to zero is not done, i.e., \bar{X} replaced by X.

Unsupervised learning

"Unsupervised learning": methods that do not exploit known labels Example of digits: perform a 2-D projection Images of same digit tend to cluster (more or less) Such 2-D representations are popular for visualization Can also try to find natural clusters in data, e.g., in materials

Basic clusterning technique: Kmeans



Example 1: The 'Swill-Roll' (2000 points in 3-D)

Original Data in 3-D



2-D 'reductions':





ONPP





Example 2: Digit images (a random sample of 30)



2-D 'reductions':



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Supervised learning: classification

Problem: Given labels (say "A" and "B") for each item of a given set, find a mechanism to classify an unlabelled item into either the "A" or the "B" class.



- Many applications.
- Example: distinguish SPAM and non-SPAM messages
- Can be extended to more than 2 classes.

Supervised learning: classification

Best illustration: written digits recognition example



Roughly speaking: we seek dimension reduction so that recognition is 'more effective' in low-dim. space

Basic method: K-nearest neighbors (KNN) classification

Idea of a voting system: get distances between test sample and training samples

• Get the k nearest neighbors (here k = 8)

Predominant class among these k items is assigned to the test sample ("*" here)



Linear classifiers and Fisher's LDA

Idea for two classes: Find a hyperplane which best separates the data in classes A and B.





Use kernels to transform



Transformed data with a Gaussian Kernel

Fisher's Linear Discriminant Analysis (LDA)

Goal: Use label information to define a good projector, i.e., one that can 'discriminate' well between given classes

Define "between scatter": a measure of how well separated two distinct classes are.

► Define "within scatter": a measure of how well clustered items of the same class are.

Objective: make "between scatter" measure large and "within scatter" small.

Idea: Find projector that maximizes the ratio of the "between scatter" measure over "within scatter" measure

$$\begin{array}{ll} \text{Define:} & \text{Where:} \\ S_B = \sum\limits_{k=1}^c n_k (\mu^{(k)} - \mu) (\mu^{(k)} - \mu)^T, \\ S_W = \sum\limits_{k=1}^c \sum\limits_{x_i \in X_k} (x_i - \mu^{(k)}) (x_i - \mu^{(k)})^T \end{array} \stackrel{\textbf{Where:}}{=} \begin{array}{l} \text{where:} \\ \textbf{Where:} \\ \textbf{Where:}$$



Consider 2nd moments for a vector a:

$$a^T S_B a \,=\, \sum\limits_{i=1}^c n_k \, |a^T (\mu^{(k)} - \mu)|^2, \ a^T S_W a \,=\, \sum\limits_{k=1}^c \sum\limits_{x_i \,\in\, X_k} |a^T (x_i - \mu^{(k)})|^2$$

 \blacktriangleright $a^T S_B a \equiv$ weighted variance of projected μ_j 's

 $\succ a^T S_W a \equiv$ w. sum of variances of projected classes X_j 's

► LDA projects the data so as to maximize the ratio of these two numbers:

 $\max_{a}rac{a^T S_B a}{a^T S_W a}$

> Optimal a = eigenvector associated with the largest eigenvalue of: $S_B u_i = \lambda_i S_W u_i$.

LDA – Extension to arbitrary dimensions

Criterion: maximize the ratio of two traces:



- > Constraint: $U^T U = I$ (orthogonal projector).
- > Reduced dimension data: $Y = U^T X$.

Common viewpoint: hard to maximize, therefore ...

In alternative: Solve instead the ('easier') problem:

 $\max_{U^T S_W U = I} ext{Tr} \left[U^T S_B U
ight]$

> Solution: largest eigenvectors of $S_B u_i = \lambda_i S_W u_i$.

LDA – Extension to arbitrary dimensions (cont.)

Consider the original problem:

$$\max_{U \in \mathbb{R}^{n \times p}, \ U^T U = I} \ \frac{\text{Tr}\left[U^T A U\right]}{\text{Tr}\left[U^T B U\right]}$$

Let A, B be symmetric & assume that B is semi-positive definite with rank(B) > n - p. Then Tr $[U^T A U]/\text{Tr} [U^T B U]$ has a finite maximum value ρ_* . The maximum is reached for a certain U_* that is unique up to unitary transforms of columns.

Consider the function:

$$f(
ho) = \max_{V^T V = I} \operatorname{Tr} \left[V^T (A -
ho B) V
ight]$$

> Call $V(\rho)$ the maximizer.

Note: $V(\rho)$ = Set of eigenvectors - not unique

 Define $G(\rho) \equiv A - \rho B$ and its *n* eigenvalues: $\mu_1(\rho) \ge \mu_2(\rho) \ge \cdots \ge \mu_n(\rho)$.
 Clearly:

$$f(
ho)=\mu_1(
ho)+\mu_2(
ho)+\dots+\mu_p(
ho)$$
 .

Can express this differently. Define eigenprojector:

 $P(
ho) = V(
ho)V(
ho)^T$



$$\begin{split} f(\rho) &= \operatorname{Tr}\left[V(\rho)^T G(\rho) V(\rho)\right] \\ &= \operatorname{Tr}\left[G(\rho) V(\rho) V(\rho)^T\right] \\ &= \operatorname{Tr}\left[G(\rho) P(\rho)\right]. \end{split}$$

Recall [e.g.
$$P(\rho) = \frac{-1}{2\pi i} \int_{\Gamma} (G(\rho) - zI)^{-1} dz$$
Kato '65] that:

 Γ is a smooth curve containing the p eivenvalues of interest

$$\textbf{ Hence: } f(\rho) = \frac{-1}{2\pi i} \text{Tr } \int_{\Gamma} G(\rho) (G(\rho) - zI)^{-1} dz = \dots \\ = \frac{-1}{2\pi i} \text{Tr } \int_{\Gamma} z (G(\rho) - zI)^{-1} dz$$

> With this, can prove :

1.
$$f$$
 is a non-increasing function of ρ ;
2. $f(\rho) = 0$ iff $\rho = \rho_*$;
3. $f'(\rho) = -\text{Tr} [V(\rho)^T B V(\rho)]$

Can now use Newton's method.

$$\rho_{new} = \rho - \frac{\operatorname{Tr} \left[V(\rho)^T (A - \rho B) V(\rho) \right]}{-\operatorname{Tr} \left[V(\rho)^T B V(\rho) \right]} = \frac{\operatorname{Tr} \left[V(\rho)^T A V(\rho) \right]}{\operatorname{Tr} \left[V(\rho)^T B V(\rho) \right]}$$

$$\blacktriangleright \text{ Newton's method to find the zero of } f \equiv \text{ a fixed point}$$

$$\operatorname{iteration with} \quad g(\rho) = \frac{\operatorname{Tr} \left[V^T(\rho) A V(\rho) \right]}{\operatorname{Tr} \left[V^T(\rho) B V(\rho) \right]},$$

Idea: Compute V(ρ) by a Lanczos-type procedure
 Note: Standard problem - [not generalized] → inexpensive!
 See T. Ngo, M. Bellalij, and Y.S. 2010 for details

Recent papers advocated similar or related techniques

[1] C. Shen, H. Li, and M. J. Brooks, A convex programming approach to the trace quotient problem. In ACCV(2) - 2007.

[2] H. Wang, S.C. Yan, D.Xu, X.O. Tang, and T. Huang. Trace ratio vs. ratio trace for dimensionality reduction. In *IEEE Conference on Computer Vision and Pattern Recognition, 2007*

[3] S. Yan and X. O. Tang, "Trace ratio revisited" Proceedings of the European Conference on Computer Vision, 2006.

. . .

Background: The Lanczos procedure

ALGORITHM : 1 . Lanczos

1. Choose an initial vector
$$v_1$$
 of norm unity.
Set $\beta_1 \equiv 0, v_0 \equiv 0$
2. For $j = 1, 2, \dots, m$ Do:
3. $\alpha_j := (w_j, v_j)$
4. $w_j := Av_j - \alpha_j v_j - \beta_j v_{j-1}$
5. $\beta_{j+1} := ||w_j||_2$. If $\beta_{j+1} = 0$ then Stop
6. $v_{j+1} := w_j/\beta_{j+1}$
7. EndDo

In first few steps of Newton: rough approximation needed.

GRAPH-BASED TECHNIQUES

Graph-based methods

Start with a graph of data. e.g.: graph of k nearest neighbors (k-NN graph)
 Want: Perform a projection which preserves the graph in some sense

Define a graph Laplacean:

$$L = D - W$$

Г

e.g.,:
$$w_{ij} = \left\{ egin{array}{ccc} 1 & ext{if} & j \in N_i \\ 0 & ext{else} \end{array} & D = ext{diag} & d_{ii} = \sum_{j
eq i} w_{ij} \end{array}
ight\}$$

with N_i = neighborhood of i (excluding i)

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Example: The Laplacean eigenmaps approach

Laplacean Eigenmaps [Belkin-Niyogi '01] *minimizes*

$$\mathcal{F}(Y) = \sum_{i,j=1}^n w_{ij} \|y_i - y_j\|^2$$
 subject to $YDY^ op = I$

Motivation: if $||x_i - x_j||$ is small (orig. data), we want $||y_i - y_j||$ to be also small (low-Dim. data) \blacktriangleright Original data used indirectly through its graph \blacktriangleright Leads to $n \times n$ sparse eigenvalue problem [In 'sample' space]



Locally Linear Embedding (Roweis-Saul-00)

Very similar to Eigenmaps.

Graph Laplacean matrix is replaced by an 'affinity' graph

Graph:Each x_i written as a convexcombination of its k nearest neighbors: $x_i \approx \Sigma w_{ij} x_j, \quad \sum_{j \in N_i} w_{ij} = 1$ \blacktriangleright Optimalweightscomputed('localcalculation')byminimizing $\|x_i - \Sigma w_{ij} x_j\|$ for $i = 1, \cdots, n$



> Mapped data (Y) computed by minimizing

$$\sum \left\|y_i - \Sigma w_{ij} y_j
ight\|^2$$

Implicit vs explicit mappings

▶ In PCA the mapping Φ from high-dimensional space (\mathbb{R}^m) to low-dimensional space (\mathbb{R}^d) is explicitly known:

$$y = \Phi(x) \equiv V^T x$$

In Eigenmaps and LLE we only know

$$y_i=\phi(x_i), i=1,\cdots,n$$

- > Not easy to get $\phi(x)$ for an arbitrary x not in the sample.
- Methods are powerful but inconvenient for classification
- "The out-of-sample extension" problem

ONPP (Kokiopoulou and YS '05)

Orthogonal Neighborhood Preserving Projections

> A linear (orthogonoal) version of LLE obtained by writing Y in the form $Y = V^{\top}X$

Same graph as LLE. Objective: preserve the affinity graph (as in LEE) *but* with the constraint $Y = V^{\top}X$

Problem solved to obtain mapping:

$$\min_{V} \operatorname{Tr} \left[V^{\top} X (I - W^{\top}) (I - W) X^{\top} V \right]$$

s.t. $V^{T} V = I$

 \blacktriangleright In LLE replace $V^{ op}X$ by Y

Face Recognition – background

Problem: We are given a database of images: [arrays of pixel values]. And a test (new) image.



Face Recognition – background

Problem: We are given a database of images: [arrays of pixel values]. And a test (new) image.





Question: Does this new image correspond to one of those in the database?

Example: Eigenfaces [Turk-Pentland, '91]

- Idea identical with the one we saw for digits:
- Consider each picture as a (1-D) column of all pixels
- Put together into an array A of size $\#_pixels \times \#_images$.



– Do an SVD of \boldsymbol{A} and perform comparison with any test image in low-dim. space

- Similar to LSI in spirit - but data is not sparse.

Graph-based methods in a supervised setting

Graph-based methods can be adapted to supervised mode. Idea: Build G so that nodes in the same class are neighbors. If c = # classes, G consists of c cliques.

- Weight matrix W = block-diagonal
 Note: rank(W) = n c.
 As before, graph Laplacean: W = $\begin{pmatrix} W_1 \\ W_2 \\ \vdots \\ U_c = D W \end{pmatrix}$
- Can be used for ONPP and other graph based methods
- Improvement: add repulsion Laplacean [Kokiopoulou, YS 09]



Leads to eigenvalue problem with matrix:

$$L_c - \rho L_R$$

•
$$L_c = class-Laplacean$$
,

•
$$L_R$$
 = repulsion Laplacean,

 $\rho = parameter$

Test: ORL 40 subjects, 10 sample images each – example:



of pixels : 112×92 ; TOT. # images : 400



> Observation: some values of ρ yield better results than using the optimum ρ obtained from maximizing trace ratio

LINEAR ALGEBRA METHODS: EXAMPLES

IR: Use of the Lanczos algorithm (J. Chen, YS '09)

Lanczos is good at catching large (and small) eigenvalues: can compute singular vectors with Lanczos, & use them in LSI

Can do better: Use the Lanczos vectors directly for the projection..

See: K. Blom and A. Ruhe [SIMAX, vol. 26, 2005]

Major difference with what we do: A Krylov subspace for each query [expensive].

Proposed: One Lanczos run- random initial vector. Then use Lanczos vectors in place of singular vectors.

Tests: IR

Information		# Terms	# Docs	# queries	sparsity
retrieval	MED	7,014	1,033	30	0.735
datasets	CRAN	3,763	1,398	225	1.412

Med dataset.





Average query times

Med dataset

Cran dataset.



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Average retrieval precision

Med dataset

Cran dataset



Retrieval precision comparisons

In summary:

- Results comparable to those of SVD ...
- .. at a much lower cost.

Thanks:

► Helpful tools and datasets widely available. We used TMG [developed at the U. of Patras (D. Zeimpekis, E. Gallopoulos)]

Updating the SVD (E. Vecharynski and YS'13)

> In applications, data matrix X often updated

Example: Information Retrieval (IR), can add documents, add terms, change weights, ..

Problem

Given the partial SVD of X, how to get a partial SVD of X_{new}

> Will illustrate only with update of the form $X_{new} = [X, D]$ (documents added in IR)

Updating the SVD: Zha-Simon algorithm

Assume $A \approx U_k \Sigma_k V_k^T$ and $A_D = [A, D]$, $D \in \mathbb{R}^{m \times p}$ Compute $D_k = (I - U_k U_k^T) D$ and its QR factorization: $[\hat{U}_p, R] = qr(D_k, 0), R \in \mathbb{R}^{p \times p}, \quad \hat{U}_p \in \mathbb{R}^{m \times p}$ Note: $A_D \approx [U_k, \hat{U}_p] H_D \begin{bmatrix} V_k & 0 \\ 0 & I_p \end{bmatrix}^T; \quad H_D \equiv \begin{bmatrix} \Sigma_k & U_k^T D \\ 0 & R \end{bmatrix}$

> Zha–Simon ('99): Compute the SVD of H_D & get approximate SVD from above equation

It turns out this is a Rayleigh-Ritz projection method for the SVD [E. Vecharynski & YS 2013]

Can show optimality properties as a result

Updating the SVD

- When the number of updates is large this becomes costly.
- > Idea: Replace \hat{U}_p by a low dimensional approximation:
- \blacktriangleright Use $ar{U}$ of the form $ar{U} = [U_k, Z_l]$ instead of $ar{U} = [U_k, \hat{U}_p]$
- > Z_l must capture the range of $D_k = (I U_k U_k^T) D$
- Simplest idea : best rank–I approximation using the SVD.

Can also use Lanczos vectors from the Golub-Kahan-Lanczos algorithm.

An example

LSI - with MEDLINE collection: m = 7,014 (terms), n = 1,033 (docs), k = 75 (dimension), t = 533 (initial # docs), $n_q = 30$ (queries)

Adding blocks of 25 docs at a time

The number of singular triplets of $(I - U_k U_k^T)D$ using SVD projection ("SV") is 2.

- For GKL approach ("GKL") 3 GKL vectors are used
- These two methods are compared to Zha-Simon ("ZS").
- We show average precision then time



Experiments show: gain in accuracy is rather consistent



Times can be significantly better for large sets

APPLICATION TO MATERIALS

Data mining for materials: Materials Informatics

- Huge potential in exploiting two trends:
- **1** Improvements in efficiency and capabilities in computational methods for materials
- **2** Recent progress in data mining techniques
- > Current practice: "One student, one alloy, one PhD" [see special MRS issue on materials informatics] \rightarrow Slow ..
- Data Mining: can help speed-up process, e.g., by exploring in smarter ways
- *Issue 1:* Who will do the work? Few researchers are familiar with both worlds

Issue 2: databases, and more generally sharing, not too common in materials

The inherently fragmented and multidisciplinary nature of the materials community poses barriers to establishing the required networks for sharing results and information. One of the largest challenges will be encouraging scientists to think of themselves not as individual researchers but as part of a powerful network collectively analyzing and using data generated by the larger community. These barriers must be overcome.

NSTC report to the White House, June 2011.

Materials genome initiative [NSF]

Unsupervised learning



1970s: Data Mining "by hand": Find coordinates that will cluster materials according to structure
 2-D projection from physical knowledge
 'Anomaly Detection': helped find that compound Cu F does not exist

see: J. R. Chelikowsky, J. C. Phillips, Phys Rev. B 19 (1978).

Question: Can modern data mining achieve a similar diagrammatic separation of structures?

- Should use only information from the two constituent atoms
- Experiment: 67 binary 'octets'.
- Use PCA exploit only data from 2 constituent atoms:
- 1. Number of valence electrons;
- 2. Ionization energies of the s-states of the ion core;
- 3. Ionization energies of the p-states of the ion core;
- 4. Radii for the s-states as determined from model potentials;
- 5. Radii for the p-states as determined from model potentials.





Supervised learning: classification

Problem: classify an unknown binary compound into its crystal structure class

- > 55 compounds, 6 crystal structure classes
- "leave-one-out" experiment

Case 1: Use features 1:5 for atom A and 2:5 for atom B. No scaling is applied.

Case 2: Features 2:5 from each atom + scale features 2 to 4 by square root of # valence electrons (feature 1)

Case 3: Features 1:5 for atom A and 2:5 for atom B. Scale features 2 and 3 by square root of # valence electrons.

Three methods tested

1. PCA classification. Project and do identification in space of reduced dimension (Euclidean distance in low-dim space).

- 2. KNN K-nearest neighbor classification -
- 3. Orthogonal Neighborhood Preserving Projection (ONPP) a graph based method [see Kokiopoulou, YS, 2005]

Recognit	ion	rates	for	3	
different	me	thods	usir	ng	
different features					

Case	KNN	ONPP	PCA
Case 1	0.909	0.945	0.945
Case 2	0.945	0.945	1.000
Case 3	0.945	0.945	0.982

Recent work

Some data is becoming available



Recent work

 Exploit Bandsctructues - in the same way we use images..
 For now we do

clustering.



▶ Work in progress
 ▶ 3-way clustering
 obtained with dim. re duction + k-means →
 ▶ Working on unrav eling the info & explor ing classification with
 the data



Conclusion

Many, interesting new matrix problems in areas that involve the effective mining of data

Among the most pressing issues is that of reducing computational cost - [SVD, SDP, ..., too costly]

Many online resources available

Huge potential in areas like materials science though inertia has to be overcome

On the + side: materials genome project is starting to energize the field To a researcher in computational linear algebra : big tide of change on types or problems, algorithms, frameworks, culture, ...

But change should be welcome:

When one door closes, another opens; but we often look so long and so regretfully upon the closed door that we do not see the one which has opened for us. Alexander Graham Bell (1847-1922)

... in the words of "Who Moved My Cheese?" [Spencer Johnson, 2002]:

"If you do not change, you can become extinct !"

"If you do not change, you can become extinct !"

"The quicker you let go of old cheese, the sooner you find new cheese."

Thank you !