Bauer-Fike Theorem

As popularized in most texts on computational linear algebra or numerical methods, the Bauer Fike Theorem is a theorem on the perturbation of eigenvalues of a diagonalizable matrix. However, it is actually just one theorem out of a small collection of theorems on the localization of eigenvalues within small regions of the complex plane [1]. In this article, we first discuss the popular theorem and then discuss a more general theorems by Bauer and Fike from which the more popular theorem was derived.

In order to state the Bauer Fike Theorem, we need the concept of a matrix norm. For the purposes of this article, we restrict our attention to "operator norms", derived from related vector norms. A vector norm is a functional $\|\cdot\|$ on a vector $\mathbf{x}$ which satisfies three properties:

1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} \equiv 0$.
2. $\|\alpha\mathbf{x} = |\alpha| \cdot \|\mathbf{x}\|$ for any scalar α.
3. $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (the triangle inequality).

A matrix norm $\|A\|$ satisfies the above three properties plus one more:

4. $\|AB\| \leq \|A\| \cdot \|B\|$ (the sub-multiplicative inequality).

An operator norm is a matrix norm derived from a related vector norm:

5. $\|A\| \stackrel{\text{def}}{=} \max_{\mathbf{x}\neq 0} \|A\mathbf{x}\| \,/\, \|\mathbf{x}\| = \max_{\|\mathbf{x}=1\|} \|A\mathbf{x}\|$.

The most common vector norms are

$$\|\mathbf{x}\|_1 = \sum_i |x_i|, \quad \|\mathbf{x}\|_2 = \left(\sum_i |x_i|^2\right)^{1/2}, \quad \|\mathbf{x}\|_\infty = \max_i |x_i|,$$

where x_i denotes the i-th entry of the vector $\mathbf{x}$. The operator norms derived from these are, respectively,

$$\|A\|_1 = \max_i \sum_j |a_{ij}|, \quad \|A\|_2 = \text{largest singular value of } A, \quad \|A\|_\infty = \max_j \sum_i |a_{ij}|,$$

where a_{ij} denotes the i,j-th entry of the matrix A. In particular, $\|A\|_\infty$ is the "maximum absolute row sum" and $\|A\|_1$ is the "maximum absolute column sum." For details, see for example [2, sec. 2.2-2.3]. Throughout this article the notation $\|\cdot\|$ will denote a vector norm when applied to a vector and an operator matrix norm when applied to a matrix.

The popular Bauer Fike Theorem involves diagonalizable matrices. Let A be an $n \times n$ real or complex matrix. The matrix A is said to be *diagonalizable* if it has a complete set of eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ corresponding to n eigenvalues $\lambda_1, \ldots, \lambda_n$. The eigenvalues need not be distinct. If we assemble the eigenvectors into an $n \times n$ matrix V, and define the diagonal matrix $\Lambda \stackrel{\text{def}}{=} \text{diag}\{\lambda_1, \ldots, \lambda_n\}$, then we have the relation $A = V\Lambda V^{-1}$.

We can now state the Bauer Fike Theorem as follows:

Theorem 1. *Let A be an $n \times n$ matrix diagonalizable satisfying $A = V\Lambda V^{-1}$, where V, Λ are defined as above, and let E be another arbitrary $n \times n$ matrix. Every eigenvalue μ of the matrix $A+E$ satisfies the inequality*

$$|\mu - \lambda| \leq \|V\| \cdot \|V^{-1}\| \cdot \|E\|, \tag{1}$$

where λ is some eigenvalue of A.

If $\|E\|$ is small, then this theorem states that each eigenvalue of $A+E$ is close to some eigenvalue of A, and that the distance between eigenvalues of $A+E$ and eigenvalues of A varies linearly with the perturbation E. But this is not an asymptotic result valid only as $E \to 0$; this result holds for *any* E. In this sense, it is a very powerful localization theorem for eigenvalues. To illustrate this point, we give two easy consequences of this theorem. Let B be an arbitrary $n \times n$ matrix, let $A \stackrel{\text{def}}{=} \text{diag}\{b_{11}, \ldots, b_{nn}\}$ be the diagonal part of B, and define $E \stackrel{\text{def}}{=} B - A$ to be the matrix of all the off-diagonal entries. Let μ be any eigenvalue of B. The matrix of eigenvectors for A is just the identity matrix, so Theorem 1 implies that $|\mu - b_{ii}| \leq \|E\|$, for some diagonal entry b_{ii}. In the ∞-norm, $\|E\|_\infty$ is just the "maximum absolute row sum over the off-diagonal elements of B." If we choose the $infty$-norm, then this means that every eigenvalue of B lies in a circle centered at a diagonal entry of B with radius equal to the "maximum absolute row sum over the off-diagonal elements." This is a weak form of a Gershgorin-type theorem which localizes the eigenvalues to circles centered at the diagonal entries, but the radius here is not as tight as in the true Gershgorin theory [2, sec. 7.2.1].

Another simple consequence of Theorem 1 applies to symmetric, Hermitian, or normal matrices. If A is symmetric, Hermitian, or normal, then the matrix V of eigenvectors can be made unitary (orthogonal if real), which means that $V^H V = I$ where V^H denotes the complex conjugate transpose of V. If we choose the 2-norm in Theorem 1, then $\|V\|_2 = \|V^{-1}\|_2 = 1$, and the inequality (1) reduces to just $|\mu - \lambda| < \|E\|$. Hence for symmetric, Hermitian, or normal matrices, the change to any eigenvalue is no larger than the norm of the change to the matrix itself.

However, Theorem 1 has some limitations. The first is that it only applies to diagonalizable matrices. The second is that it says nothing about the correspondence between an eigenvalue μ of the perturbed matrix $A+E$ and the eigenvalues of the unperturbed matrix A. In particular, as E increases from zero, it is difficult to predict how each eigenvalue of $A+E$ will move, and there is no way to predict which eigenvalue of the original A corresponds to any particular eigenvalue μ of $A+E$, except in certain special cases such as symmetric or Hermitian matrices.

However, in their original paper [1] Bauer and Fike presented several more general results to relieve the limitation to diagonalizable matrices. The most important of these is the following, for which the proof is so simple we present it as well:

Theorem 2. *Let A be an arbtrary $n \times n$ matrix, and let μ denote any eigenvalue of $A+E$. Then either μ is also an eigenvalue of A or*

$$1 \leq \|(\mu I - A)^{-1} \cdot E\| \leq \|(\mu I - A)^{-1}\| \cdot \|E\|. \tag{2}$$

Proof: This result follows immediately from a simple matrix manipulation using the norm properties given above:

$$\begin{aligned}
&(A+E)\mathbf{x} = \mu\mathbf{x} \\
&E\mathbf{x} = (\mu I - A)\mathbf{x} \\
&(\mu I - A)^{-1} \cdot E\mathbf{x} = x \\
&\|(\mu I - A)^{-1} \cdot E\| \cdot \|\mathbf{x}\| \geq \|x\|
\end{aligned}$$

□

Even though this may appear to be a more technical result, this actually leads to a great many often used results. For example, in the case $A = V\Lambda V^{-1}$ is diagonalizable (using the notation of Theorem 1), it is easy to show that

$$\|(\mu I - A)^{-1}\| = \|V(\mu I - \Lambda)^{-1} V^{-1}\| \leq \|V\| \cdot \|(\mu I - \Lambda)^{-1}\| \cdot \|V^{-1}\|$$

leading immediately to Theorem 1, where we have used the fact that the operator norm of a diagonal matrix is just its largest entry (in absolute value). We can also repeat our construction in which B is an arbitrary matrix, A is the matrix consisting of the diagonal elements of B, and $E \stackrel{\text{def}}{=} B - A$ is the matrix of off-diagonals. Using the ∞-norm, the leftmost inequality in (2) then reduces to

$$1 \leq \max_i \left(\frac{1}{|\mu - b_{ii}|} \cdot \sum_{j \neq i} |b_{ij}| \right),$$

leading immediately to the first Gershgorin theorem (q.v.) (see also [2, sec. 7.2.1]).

References

[1] F. L. Bauer and C. T. Fike. Norms and exclusion theorems. *Numerische Matematik*, 2:137–141, 1960.

[2] G. H. Golub and C. F. Van Loan. *Matrix Computations.* Johns Hopkins Univ. Press, 3rd edition, 1996.