A General Vandermonde Factorization of a Hankel Matrix*

Daniel Boley[1], Franklin Luk[2], David Vandevoorde[2]

[1] University of Minnesota, Minneapolis MN 55455, USA
[2] Rensselaer Polytechnic Institute, Troy NY 12180, USA

Abstract. It is shown that an infinite Hankel matrix of a finite rank (or a finite Hankel matrix) admits a generalized Vandermonde decomposition $H = V^T DV$, where V is a generalized Vandermonde matrix, and D is a block diagonal matrix. The full structure of this decomposition was first fully discussed by Vandevoorde [9], but the development here is based solely on linear algebra considerations, specifically the use of the Jordan Canonical Form.

1 Introduction

We consider an infinite-dimensional Hankel matrix of a finite rank r

$$H_\infty = \begin{pmatrix} h_0 & h_1 & h_2 & h_3 & \cdot^{\cdot^{\cdot}} \\ h_1 & h_2 & h_3 & h_4 & \cdot^{\cdot^{\cdot}} \\ h_2 & h_3 & h_4 & h_5 & \cdot^{\cdot^{\cdot}} \\ h_3 & h_4 & h_5 & h_6 & \cdot^{\cdot^{\cdot}} \\ \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} \end{pmatrix}, \tag{1}$$

where the entries $\{h_k\}_{k=0}^{\infty}$ are complex-valued scalars. In signal processing, these scalars represent a signal generated by a sum of a finite number of exponentials

$$h_k = \sum_{l=1}^{r} \lambda_l^k d_l, \qquad \text{for } k = 0, 1, 2, \cdots. \tag{2}$$

The goal is to find the underlying modes λ_l and weights d_l corresponding to this signal, when the signal is corrupted by noise. In the theory of orthogonal polynomials, the entries h_l represent moments with respect to an inner product or pseudo inner product. If this pseudo inner product is a weighted sum over a finite set of knots: $\langle f, g\rangle = \sum_{l=1}^{r} \overline{f}(\lambda_l) g(\lambda_l) d_l$, then $h_k = \langle 1, x^k\rangle$. The goal in this case is to determine the knots and weights of the inner product from the moments.

If all the modes λ_l are distinct, then equation (2) is equivalent to the Vandermonde decomposition of the original Hankel matrix: $H_\infty = V_\infty^T D V_\infty$ where

* This research was partially supported by NSF Grant CCR-9405380.

$D \triangleq \mathrm{diag}(d_1, \cdots, d_r)$ and

$$V_\infty \triangleq \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots \\ 1 & \lambda_2 & \lambda_2^2 & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ 1 & \lambda_r & \lambda_r^2 & \cdots \end{pmatrix}_{r\times\infty} \tag{3}$$

In this paper, we derive the general structure of the Vandermonde decomposition of H_∞ in the case of multiple modes using only linear algebra considerations, and discuss the case of finite dimensional Hankel matrices. We begin with some preliminaries, then show how the general structure of the decomposition can be derived using the Jordan canonical form, and conclude with a discussion of the finite dimensional case.

Regarding infinite Hankel matrices of finite rank, we have the following theorem from Gantmacher [4, vol. 2, p207]:

Theorem 1. *Let H_∞ be an infinite Hankel matrix of rank r. Then the entries of H_∞ satisfy an r-term recurrence of the form (for $k = r, r+1, \cdots$):*

$$h_k = a_{r-1}h_{k-1} + a_{r-2}h_{k-2} + \cdots + a_0 h_{k-r}. \tag{4}$$

Furthermore, the modes λ_l generating the entries in (2) are the roots of the polynomial of degree r:

$$p(\lambda) \triangleq \lambda^r - a_{r-1}\lambda^{r-1} - \cdots - a_0\lambda^0, \tag{5}$$

and the leading $r \times r$ principal submatrix of H_∞ has full rank r. □

It is also well known that (4) is a difference equation whose general solution $[h_0, h_1, h_2, \cdots]$ is a linear combination of solutions of the form [6, p. 33]:

$$\begin{array}{rl} (1 \;\; \lambda_l \;\; \lambda_l^2 \;\; \lambda_l^3 \;\; \cdots) &= (1 \;\; \lambda_l \;\; \lambda_l^2 \;\; \lambda_l^3 \;\; \cdots) \\ \frac{\mathrm{d}}{\mathrm{d}\lambda_l}(1 \;\; \lambda_l \;\; \lambda_l^2 \;\; \lambda_l^3 \;\; \cdots) &= (0 \;\; 1 \;\; 2\lambda_l \;\; 3\lambda_l^2 \;\; \cdots) \\ \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}\lambda_l^2}(1 \;\; \lambda_l \;\; \lambda_l^2 \;\; \lambda_l^3 \;\; \cdots) &= (0 \;\; 0 \;\; 1 \;\; 3\lambda_l \;\; \cdots) \\ \vdots & \quad\quad \vdots \\ \frac{1}{(m_l-1)!}\frac{\mathrm{d}^{m_l-1}}{\mathrm{d}\lambda_l^{m_l-1}}(1 \;\; \lambda_l \;\; \lambda_l^2 \;\; \lambda_l^3 \;\; \cdots) &= (0 \;\; \cdots \;\; 0 \;\; 1 \;\; \binom{m_l}{1}\lambda_l \;\; \cdots), \end{array} \tag{6}$$

where m_l is the multiplicity of λ_l as a root of (5). In (6), the general $(j+1)$-st entry in the $(i+1)$-st solution (for $i, j = 0, 1, \cdots$) is

$$\frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}\lambda_l^i}\lambda_l^j = \begin{cases} 0 & \text{if } i > j \\ \binom{j}{i}\lambda_l^{j-i} & \text{if } i \leq j, \end{cases}$$

where $\binom{j}{i} = \frac{j!}{i!(j-i)!}$ is the binomial coefficient. This naturally leads one to expect that the generalization of the Vandermonde decomposition $H_\infty = V_\infty^T D V_\infty$ will consist of a confluent Vandermonde matrix [5] whose rows have the form in (6). That still leaves open the structure arising in D. This we develop in the next section.

2 General Vandermonde Decomposition

Assume the matrix H_∞ of equ. (1) has rank r. Let H_r consist of the first r rows of H_∞, which from Theorem 1 has full rank r. Denote the first column of H_r by $\mathbf{h}_0 \triangleq (\, h_0 \quad \cdots \quad h_{r-1}\,)^T$. Then the difference equation (4) can be written in matrix form

$$H_r = (\, \mathbf{h}_0 \quad C\mathbf{h}_0 \quad C^2\mathbf{h}_0 \quad \cdots)_{(r\times\infty)}$$

where

$$C \triangleq \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{r-2} & a_{r-1} \end{pmatrix}$$

is the companion matrix corresponding to the polynomial (5). The eigenvalues of C are the roots of (5), and we denote them by λ_l, each of multiplicity m_l, for $l = 1, \cdots, s$, where $m_1 + \cdots + m_s = r$. The Jordan form of C will have the form $J = \mathrm{diag}\{J_1, \cdots, J_s\}$ where J_l is shorthand for $J_l = J_{m_l\times m_l}(\lambda_l)$ (viz. the $m_l \times m_l$ Jordan block corresponding to eigenvalue λ_l).

Define

$$V \triangleq (\, \mathbf{v} \quad J^T\mathbf{v} \quad (J^T)^2\mathbf{v} \quad \cdots \quad (J^T)^{r-1}\mathbf{v}\,),$$

where $\mathbf{v}^T = \left(\mathbf{e}_1^{[m_1]\,T} \quad \cdots \quad \mathbf{e}_1^{[m_s]\,T}\right)$ is a vector partitioned conformally with J and $\mathbf{e}_1^{[m_l]} = (\,1 \quad 0 \quad \cdots \quad 0\,)^T$ is the m_l dimensional unit coordinate vector. For example, if the roots were 2, 3, 1, with multiplicities 1, 1, 3, respectively, then $J, \mathbf{v}, V$ would be

$$J = \left(\begin{array}{c|c|ccc} 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right), \quad \mathbf{v} = \left(\begin{array}{c} 1 \\ \hline 1 \\ \hline 1 \\ 0 \\ 0 \end{array}\right), \quad V = \left(\begin{array}{ccccc} 1 & 2 & 4 & 8 & 16 \\ \hline 1 & 3 & 9 & 27 & 81 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \end{array}\right).$$

Then we have by the definition of V:

$$\begin{aligned} VC^T &= (\, \mathbf{v} \quad J^T\mathbf{v} \quad (J^T)^2\mathbf{v} \quad \cdots \quad (J^T)^{r-1}\mathbf{v}\,)\, C^T \\ &= (\, J^T\mathbf{v}, \quad \cdots, \quad (J^T)^{r-1}\mathbf{v}, \quad (a_0 I + \cdots + a_{r-1}J^{r-1})^T\mathbf{v}\,) \\ &= (\, J^T\mathbf{v}, \quad \cdots, \quad (J^T)^{r-1}\mathbf{v}, \quad (J^T)^r\mathbf{v}\,) \qquad (*) \\ &= J^T V, \end{aligned} \tag{7}$$

where the line (*) follows from the fact that J satisfies the characteristic equation of C: $p(J) = 0$ (the Cayley-Hamilton Theorem). Thus we have the Jordan decomposition of $C = V^T J V^{-T}$. It is necessary to justify the nonsingularity of V; one way is to note that the λ_l's are distinct and each row of V is one of the general solutions (6) to the difference equation (4), and hence V has r linearly independent rows.

We can now express H_r in terms of the Jordan canonical decomposition $CV^T = V^T J$:

$$\begin{aligned} H_r &= (\,\mathbf{h}_0 \quad C\mathbf{h}_0 \quad C^2\mathbf{h}_0 \quad \cdots\,) \\ &= V^T (\,\mathbf{w} \quad J\mathbf{w} \quad J^2\mathbf{w} \quad \cdots\,), \end{aligned} \tag{8}$$

where $\mathbf{w} \triangleq V^{-T}\mathbf{h}_0$. We claim that there exists a block diagonal matrix $D = \mathrm{diag}\{D_1, \cdots, D_s\}$ [partitioned conformally with J] satisfying the two conditions

$$\begin{aligned} &\text{(a) } D\mathbf{v} = \mathbf{w} \\ &\text{(b) } DJ^T = JD, \end{aligned} \tag{9}$$

so that

$$\begin{aligned} H_{r\times r} &= V^T (\,D\mathbf{v} \quad JD\mathbf{v} \quad J^2D\mathbf{v} \quad \cdots \quad J^{r-1}D\mathbf{v}\,) \\ &= V^T D (\,\mathbf{v} \quad J^T\mathbf{v} \quad (J^T)^2\mathbf{v} \quad \cdots \quad (J^T)^{r-1}\mathbf{v}\,) \\ &= V^T D V. \end{aligned} \tag{10}$$

Here $H_{r\times r}$ denotes the leading $r \times r$ principal submatrix of H_∞.

We now construct the matrix D to satisfy the conditions (9). Since D and J are both block diagonal with conformal blocks, and also conformal with the partitioning of $\mathbf{v}$, it suffices to look at the l-th block D_l, for each $l = 1, \cdots, s$. We partition $\mathbf{w}^T = (\,\mathbf{w}_1 \quad \cdots \quad \mathbf{w}_s\,)$ conformal with $\mathbf{v}$ and J. The condition (9a) determines the first column of each D_l: $D_l \mathbf{e}_1^{[m_l]} = \mathbf{w}_l$. The condition (9b) means that $D_l J_l^T = J_l D_l$, which is equivalent to

$$D_l (J_l - \lambda_l I)^T = (J_l - \lambda_l I) D_l. \tag{11}$$

But the matrix $(J_l - \lambda_l I)$ has the special form of a nilpotent "shift-up" matrix N whose (i, j)-th entry is given by

$$[J_l - \lambda_l I]_{i,j} = N_{i,j} = \begin{cases} 1 & \text{if } i = j - 1 \\ 0 & \text{otherwise} \end{cases}, \tag{12}$$

for $i, j = 2, \cdots, m_l$. This means that $[D_l]_{i,j-1} = [D_l]_{i-1,j}$ for $i, j = 1, \cdots, m_l$. Furthermore, the last row of (11) [right side] is zero, so $[D_l]_{i,j} = 0$ for $i = m_l$ and $j = 2, \cdots, m_l$. So we conclude that the conditions $D\mathbf{v} = \mathbf{w}$ and $DJ^T = JD$ imply that D_l is upper antitriangular and Hankel, and furthermore such a D_l satisfying these two conditions exists and is uniquely defined by these two conditions. This can be illustrated with an example representing a 3×3 D_l:

$$\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix}$$

which means that

$$\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} = \begin{pmatrix} D_1 & D_2 & D_3 \\ D_2 & D_3 & 0 \\ D_3 & 0 & 0 \end{pmatrix}$$

is Hankel and upper anti-triangular.

Hence we have the following:

Theorem 2. *Suppose H_∞ is an infinite matrix of rank r, and let $\lambda_1, \cdots, \lambda_s$ be the roots of the r-term difference equation generating the entries of H_∞, where each λ_l has multiplicity m_l, $l = 1, \cdots, s$. Then H_∞ admits a generalized Vandermonde decomposition $H_\infty = V_\infty^T D V_\infty$ where V_∞ is the $r \times \infty$ confluent Vandermonde matrix*

$$V_\infty = \begin{pmatrix} V_\infty^{[1]} \\ \vdots \\ V_\infty^{[s]} \end{pmatrix}_{r\times\infty}$$

where each block row $V_\infty^{[l]}$ has m_l rows with the form

$$V_\infty^{[l]} = \begin{pmatrix} 1 & \lambda_l & \lambda_l^2 & \lambda_l^3 & \lambda_l^4 & \cdots & \lambda_l^{m_l-1} & \cdots \\ 0 & 1 & 2\lambda_l & 3\lambda_l^2 & 4\lambda_l^3 & \cdots & \binom{m_l-1}{1}\lambda_l^{m_l-2} & \cdots \\ 0 & 0 & 1 & 3\lambda_l & 6\lambda_l^2 & \cdots & \binom{m_l-1}{2}\lambda_l^{m_l-3} & \cdots \\ 0 & 0 & 0 & 1 & 4\lambda_l^1 & \cdots & \binom{m_l-1}{3}\lambda_l^{m_l-4} & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots & \binom{m_l-1}{4}\lambda_l^{m_l-5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & \cdots \end{pmatrix}_{m_l\times\infty}, \qquad (13)$$

and D is a block diagonal matrix whose l-th diagonal block is $m_l \times m_l$ (partitioned conformally with V) and is Hankel and upper anti-triangular. The $(i+1, j+1)$-st entry of $V_\infty^{[l]}$ is given by

$$[V_\infty^{[l]}]_{i+1,j+1} = \begin{cases} 0 & \text{if } i > j \\ \binom{j}{i}\lambda^{j-i} & \text{if } i \le j, \end{cases}$$

for $i = 0, \cdots, m_l - 1$ and $j = 0, 1, \cdots$.

Proof: The decomposition of the leading $r \times r$ principal submatrix $H_{r\times r} = V^T D V$ where $V \triangleq (\mathbf{v} \quad J^T\mathbf{v} \quad (J^T)^2\mathbf{v} \quad \cdots \quad (J^T)^{r-1}\mathbf{v})$, follows from the above discussion. Defining the $r \times \infty$ matrix $V_\infty \triangleq (\mathbf{v} \quad J^T\mathbf{v} \quad (J^T)^2\mathbf{v} \quad \cdots)$, equation (8) leads to:

$$\begin{aligned} H_r &= V^T (D\mathbf{v} \quad JD\mathbf{v} \quad J^2D\mathbf{v} \quad \cdots) \\ &= V^T (D\mathbf{v} \quad DJ^T\mathbf{v} \quad D(J^T)^2\mathbf{v} \quad \cdots) \\ &= V^T D V_\infty. \end{aligned}$$

Because of the symmetry of H_∞, we have

$$H_\infty = \begin{pmatrix} I \\ (C^T)^r \\ (C^T)^{2r} \\ \vdots \end{pmatrix} V^T D V_\infty = \begin{pmatrix} V^T \\ V^T J^r \\ V^T J^{2r} \\ \vdots \end{pmatrix} D V_\infty = V_\infty T D V_\infty.$$

The specific structure of each block row $V_\infty^{[l]}$ can be derived from the following. The $(j+1)$-st column of $V_\infty^{[l]}$ is $[V_\infty^{[l]}]_{:,j+1} = (J_l^T)^j \mathbf{e}_1^{[m_l]}$ where

$$(J_l^T)^j = \begin{pmatrix} \lambda_l^j & & & & \\ \binom{j}{1}\lambda_l^{j-1} & \lambda_l^j & & 0 & \\ \binom{j}{2}\lambda_l^{j-2} & \binom{j}{1}\lambda_l^{j-1} & \lambda_l^j & & \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{j}{m_l-1}\lambda_l^{j-m_l+1} & \binom{j}{m_l-2}\lambda_l^{j-m_l+2} & \binom{j}{m_l-3}\lambda_l^{j-m_l+3} & \cdots & \lambda_l^j \end{pmatrix}.$$

and $\mathbf{e}_1^{[m_l]} = (1 \quad 0 \quad 0 \quad \cdots \quad 0)^T$ (an m_l-vector), where if $\lambda_j = 0$ we use the convention that $0^0 = 1$, $0^k = 0$ for any $k \neq 0$. The formula for the powers of the Jordan block $(J_l^T)^j$ can be verified by applying the binomial theorem to $J_l^j = (N + \lambda_l I)^j = \sum_i \binom{j}{i} N^i \lambda_l^{j-i}$, where N is the nilpotent matrix (12). □

3 Finite Dimensional Hankel Matrix

Let $H_{r\times r}$ be an order r nonsingular Hankel matrix:

$$H_{r\times r} = \begin{pmatrix} h_0 & h_1 & h_2 & \cdots & h_{r-1} \\ h_1 & h_2 & h_3 & \cdots & h_r \\ \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ h_{r-2} & h_{r-1} & h_r & \cdots & h_{2r-3} \\ h_{r-1} & h_r & h_{r+1} & \cdots & h_{2r-2} \end{pmatrix}$$

If we can find an r term recurrence of the form (2) generating the h_k, then we have the Vandermonde decomposition (10) for this finite dimensional Hankel matrix. Any such recurrence can be completely determined by choosing an extra parameter $\gamma = h_{2r-1}$ by means of the "Yule-Walker" equations (originally found in Prony [8]):

$$\begin{pmatrix} h_0 & h_1 & h_2 & \cdots & h_{r-1} \\ h_1 & h_2 & h_3 & \cdots & h_r \\ \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ h_{r-2} & h_{r-1} & h_r & \cdots & h_{2r-3} \\ h_{r-1} & h_r & h_{r+1} & \cdots & h_{2r-2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{r-1} \end{pmatrix} = \begin{pmatrix} h_r \\ h_{r+1} \\ \vdots \\ h_{2r-2} \\ \gamma \end{pmatrix} \tag{14}$$

for the coefficients of the r term recurrence. Since $H_{r\times r}$ is assumed to be nonsingular, we find that there is a one-parameter family of r-term recurrence relations and hence a one-parameter family of $r\times r$ Vandermonde decompositions. A further analysis of this family is as follows. Consider the system of equations

$$\begin{pmatrix} h_0 & h_1 & h_2 & \cdots & h_{r-1} \\ h_1 & h_2 & h_3 & \cdots & h_r \\ \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ h_{r-2} & h_{r-1} & h_r & \cdots & h_{2r-3} \\ h_{r-1} & h_r & h_{r+1} & \cdots & h_{2r-2} \end{pmatrix} \begin{pmatrix} b_0 & c_0 \\ b_1 & c_1 \\ b_2 & c_2 \\ \vdots & \vdots \\ b_{r-1} & c_{r-1} \end{pmatrix} = \begin{pmatrix} h_r & 0 \\ h_{r+1} & 0 \\ \vdots & \vdots \\ h_{2r-2} & 0 \\ 0 & 1 \end{pmatrix} \tag{15}$$

and define the two polynomials (not identically zero)

$$b(x) = x^r - b_{r-1}x^{r-1} - \cdots - b_0$$

and

$$c(x) = -c_{r-1}x^{r-1} - \cdots - c_0.$$

Then the polymomial $p(x)$ of equ. (5) corresponding to the solution of (14) can be written as $p(x) = b(x) + \gamma c(x)$, where γ denotes a free parameter. Suppose we are given a value λ to which we wish to set one root of $p(x)$. If $c(\lambda) \neq 0$, we can achieve this by setting $\gamma := -b(\lambda)/c(\lambda)$. But $c(\lambda) = 0$ only at up to $r-1$ distinct locations. So we have proved the following result.

Theorem 3. *Let $H_{r\times r}$ be a nonsingular $r \times r$ Hankel matrix. Then $H_{r\times r}$ admits a family of $r \times r$ Vandermonde decompositions of the form (10) parametrized by a single complex-valued parameter $\gamma = h_{2r-1}$ which can be chosen arbitrarily. Furthermore, given any complex λ (except for up to $r-1$ isolated points in the complex plane) there exists a Vandermonde decomposition such that one of the block rows (13) of the Vandermonde matrix is generated by λ (i.e., λ is a root of the polynomial (5)).* □

We illustrate this result with the following example, where the roots are 2, 3, 1 with respective multiplicities 1, 1, 3:

$$\begin{pmatrix} 6 & 8 & 17 & 43 & 114 \\ 8 & 17 & 43 & 114 & 310 \\ 17 & 43 & 114 & 310 & 863 \\ 43 & 114 & 310 & 863 & 2453 \\ 114 & 310 & 863 & 2453 & 7088 \end{pmatrix} = V^T D V =$$

$$= \left(\begin{array}{r|r|rrr} 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ 4 & 9 & 1 & 2 & 1 \\ 8 & 27 & 1 & 3 & 3 \\ 16 & 81 & 1 & 4 & 6 \end{array}\right) \left(\begin{array}{c|c|ccc} 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \left(\begin{array}{rrrrr} 1 & 2 & 4 & 8 & 16 \\ \hline 1 & 3 & 9 & 27 & 81 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \end{array}\right)$$

Now we briefly consider the case of an $n \times n$ Hankel matrix which is singular. In this case we can embed this matrix in an $r \times r$ nonsingular Hankel matrix. This can be done by extending the "signal" $\{h_k\}_{k=0}^{2n-2}$. There are many choices for such an extension, and an open question is how to choose the extension to obtain the smallest possible rank $r_{\min}$ that suffices to construct such an embedding. We remark that we use the term embedding loosely, since it is possible that $r < n$, which occurs when the leading $r \times r$ principal submatrix of $H_{n\times n}$ has full rank $r = \text{rank}\{H_{n\times n}\}$. In fact, we have the following result

Theorem 4. *If the leading $r \times r$ principal submatrix of an $n \times n$ Hankel matrix H is nonsingular, where r is the rank of H, then H admits the Vandermonde decomposition $H = (V_{r\times n})^T D_{r\times r} V_{r\times n}$, where the matrices have the indicated dimensions.*

Proof: Let $H_{r\times r}$ denote the leading $r \times r$ principal submatrix, and let $\gamma = H_{r,r+1} = H_{r+1,r} = h_{2r-1}$. It suffices to show that all the entries $\{h_k\}_{k=0}^{2n-2}$ satisfy the r-th order difference equation (4) for $k = r, \ldots, 2n-2$. Because then it would follow from Theorem 1 that H is the leading principal submatrix of an infinite dimensional Hankel matrix H_∞ of rank r, and hence we could obtain the Vandermonde decomposition of H as the leading part of the decomposition of H_∞.

Let $\mathbf{h}_{k-1}$ denote the k-th column of H, as an extension to the notation $\mathbf{h}_0$ denoting the first column of H. The first r columns are linearly independent but not the first $r+1$ columns, so

$$\mathbf{h}_k = a_{r-1}\mathbf{h}_{k-1} + a_{r-2}\mathbf{h}_{k-2} + \cdots + a_0\mathbf{h}_{k-r} \tag{16}$$

holds for $k = r$, where the a's are the solution to (14). We claim that (16) holds also for $k = r+1$: certainly this is true for the first $n-1$ rows of (16) because of the Hankel structure. Since the last row is a linear combination of the first $n-1$, the linear relation (16) must also hold for the last row. Repeating the same argument, (16) must hold for $k = r+2, r+3, \ldots, n-1$. This implies that the difference equation (4) must apply for $k = r, r+1 \ldots, 2n-2$. □

We can conclude that the length $r_{\min}$ of the shortest recurrence generating the entries of H (and hence the order of the resulting Vandermonde decomposition) will be less than or equal to n if r is the rank of H and the leading $r \times r$ principal submatrix of H is nonsingular. Otherwise $r > n$. We illustrate these situations with the following:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ has } r_{\min} = 4, \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ has } r_{\min} = 2.$$

4 Smallest Rank One Perturbation To Reduce Rank

As an illustration of the theory presented above, we address the question of finding the smallest rank-one change to a nonsingular Hankel matrix that will reduce its rank. Specifically, we address the following problem:

Problem P1. *Given an $r \times r$ non-singular Hankel matrix H, find a Hankel matrix $\widetilde{H}$ such that rank$(\widetilde{H}) = r-1$, rank$(H - \widetilde{H}) = 1$, and $\|H - \widetilde{H}\|$ is a minimum.*

We remark that this problem differs from that of finding the nearby Hankel matrix of lower rank by means of the algorithm of Cadzow [3] in that the difference matrix $H - \widetilde{H}$ may not be rank one, and generally has full rank.

Suppose there is a solution to Problem P1. Since $\widehat{H} \triangleq (H - \widetilde{H})$ is also a Hankel matrix, it must admit a Vandermonde decomposition, and since $\widehat{H}$ has rank one, this Vandermonde decomposition must take on a very special form: $\widehat{H} = \widehat{V}^T \widehat{d} \widehat{V}$ where $\widehat{V} = \begin{pmatrix} 1 & \widehat{\lambda} & \widehat{\lambda}^2 & \cdots & \widehat{\lambda}^{r-1} \end{pmatrix}$ consists of a single row generated by $\widehat{\lambda}$, and $\widehat{d}$ is a scalar.

Now consider the polynomials $b(x)$, $c(x)$ derived from the system (15). If $c(\widehat{\lambda}) \neq 0$, we can find a unique value for the scalar γ such that $p(\widehat{\lambda}) = b(\widehat{\lambda}) + \gamma c(\widehat{\lambda})$, which leads to a Vandermonde decomposition of the original matrix H in which $\widehat{\lambda}$ appears as one of the roots:

$$H = V^T D V = \begin{pmatrix} & 1 & \\ & \widehat{\lambda} & \\ \cdots & & \cdots \\ & \vdots & \\ & \widehat{\lambda}^{r-2} & \\ & \widehat{\lambda}^{r-1} & \end{pmatrix} \begin{pmatrix} \ddots & & \\ & d & \\ & & \ddots \end{pmatrix} \begin{pmatrix} & & & \vdots & \\ 1 & \widehat{\lambda} & \cdots & \widehat{\lambda}^{r-2} & \widehat{\lambda}^{r-1} \\ & & & \vdots & \end{pmatrix}. \tag{17}$$

We can write the rank-one change as $\widehat{H} = V^T \widehat{D} V$, using the same V as in (17) and with $\widehat{D} = \mathrm{diag}\{0, \ldots, 0, \widehat{d}, 0 \ldots, 0\}$. Then the singularity of

$$\widetilde{H} = H - \widehat{H} = V^T (D - \widehat{D}) V$$

implies that $\widehat{\lambda}$ must correspond to the scalar diagonal block $d = \widehat{d}$ of D (i.e. it must be a simple root of (5)). Therefore, the decomposition of the reduced-rank matrix $\widetilde{H}$ will consist of the other $r-1$ rows of V and the corresponding $r-1$ rows and columns of D, derived from the roots other than $\widehat{\lambda}$. This in turn implies that the leading $(r-1) \times (r-1)$ principal submatrix of $\widetilde{H}$ is nonsingular, by Theorem 2.

Conversely, let $\widetilde{H}$ be the solution to problem P1 and suppose that the leading $(r-1) \times (r-1)$ principal submatrix of $\widetilde{H}$ is nonsingular. Then we can obtain the Vandermonde decomposition of $H = \widetilde{H} + \widehat{H}$ by just taking the direct sum of the decompositions of $\widetilde{H}$ and $\widehat{H}$. Specifically: if $\widetilde{H} = \widetilde{V}^T \widetilde{D} \widetilde{V}$ and $\widehat{H} = \widehat{V}^T \widehat{d} \widehat{V}$ are Vandermonde decompositions of order $r-1$ and 1, respectively, then

$$H = \begin{pmatrix} \widetilde{V}^T \\ \widehat{V}^T \end{pmatrix} \begin{pmatrix} \widetilde{D} & \\ & \widehat{d} \end{pmatrix} \begin{pmatrix} \widetilde{V} & \widehat{V} \end{pmatrix} \tag{18}$$

is a Vandermonde decomposition of the original H of order r.

So we have demonstrated that either there is a value of the parameter γ which will produce a Vandermonde decomposition (18) from which a solution to Problem P1 may be extracted, or the solution to Problem P1 will either fail to exist or will have a singular leading $(r-1) \times (r-1)$ principal submatrix.

5 Conclusions

We have derived the general form of a Vandermonde decomposition for an infinite Hankel matrix of finite rank and of a finite dimensional nonsingular Hankel matrix. We have discussed some choices when decomposing a singular finite-dimensional Hankel matrix, and showed how this theory leads to a partial solution to the "smallest rank-one change" problem. These latter two situations deserve further study and analysis. A later paper will report on a fast algorithm to compute this decomposition which has been developed by Vandevoorde [9] and briefly described in [1], based on the Lanczos techniques in [7, 2].

References

1. D. Boley, F. Luk, and D. Vandevoorde. Factorization of hankel matrices. In *Workshop on Scientific Computing 97*, March 1997. Hong Kong, (to appear).
2. D. L. Boley, T. J. Lee, and F. T. Luk. The Lanczos algorithm and Hankel matrix factorization. *Lin. Alg. & Appl.*, 172:109–133, 1992.
3. J. Cadzow. Signal enhancement – a composite property mapping algorithm. *IEEE Trans. Acoust., Speech., Sig. Proc.*, 36:49–62, 1988.
4. F. R. Gantmacher. *Theory of Matrices.* Chelsea, New York, 1959.
5. G. H. Golub and C. F. Van Loan. *Matrix Computations.* Johns Hopkins Univ. Press, 3rd edition, 1996.
6. D. Kincaid and W. Cheney. *Numerical Analysis.* Brooks/Cole, 2nd edition, 1996.
7. J. L. Phillips. The triangular decomposition of Hankel matrices. *Math. Comp.*, 25(115):599–602, 1971.
8. R. Prony. Essai expérimental et analytique sur les lois de la dilatabilité et sur celles de la force expansive de la vapeur de l'eau et de la vapeur de l'alkool, à différentes températures. *J. de l'École Polytechnique*, 1:24–76, 1795.
9. D. Vandevoorde. *A Fast Exponential Decomposition Algorithm and its Applications to Structured Matrices.* PhD thesis, Computer Science Department, Rensselaer Polytechnic Institute, 1996.

This article was processed using the LaTeX macro package with LLNCS style