

Sparse Triangular Systems

- **Triangular systems**
- **Sparse triangular systems with dense right-hand sides**
- **Sparse triangular systems with sparse right-hand sides**
- **A sparse factorization based on sparse triangular solves**

Sparse Triangular linear systems: the problem

The Problem: A is an $n \times n$ matrix, and b a vector of \mathbb{R}^n . Find x such that:

$$Ax = b$$

- x is the unknown vector, b the right-hand side, and A is the coefficient matrix
- We consider the case when A is upper (or lower)triangular.

Two cases:

1. A sparse, b dense vector [solve once or many times]
2. A sparse, b sparse vector [solve once or many times]

Triangular linear systems

Example:

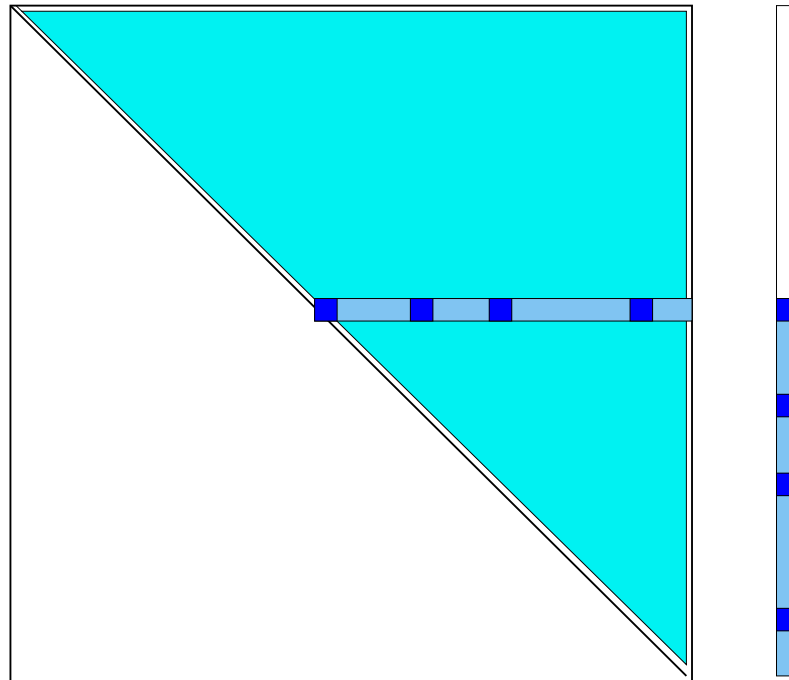
$$\begin{bmatrix} 2 & 4 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

Back-Substitution
Row version

```
For  $i = n : -1 : 1$  do:  
   $t := b_i$   
  For  $j = i + 1 : n$  do  
     $t := t - a_{ij}x_j$   
  End  
   $x_i = t/a_{ii}$   
End
```

 1 Operation count?

Illustration for sparse case (Sparse A , dense b)



- This will use the CSR data structure
- Inner product of a sparse row with a dense column
- Sparse BLAS: Sparse 'sdot'

➤ Recall:

```
typedef struct SpaFmt {
/*-----
| C-style CSR format - used internally
| for all matrices in CSR format
|-----*/
    int n;
    int *nzcount; /* length of each row */
    int **ja;     /* to store column indices */
    double **ma; /* to store nonzero entries */
} CsMat, *csptr;
```

- Can store rows of a matrix (CSR) or its columns (CSC)
- For triangular systems that are solved once, or many times with same matrix, we will assume that diagonal entry is stored in first location in inverted form.
- Result:

```

void Usol(csptr mata, double *b, double *x)
{
    int i, k, *ki;
    double *ma;
    for (i=mata->n-1; i>=0; i--) {
        ma = mata->ma[i];
        ki = mata->ja[i];
        x[i] = b[i] ;
        // Note: diag. entry avoided
        for (k=1; k<mata->nzcount[i]; k++)
            x[i] -= ma[k] * x[ki[k]];
        x[i] *= ma[0];
    }
}

```

➤ Operation count?

Column version

➤ Column version of back-substitution:

Back-Substitution
Column version

```
For  $j = n : -1 : 1$  do:  
   $x_j = b_j / a_{jj}$   
  For  $i = 1 : j - 1$  do  
     $b_i := b_i - x_j * a_{ij}$   
  End  
End
```


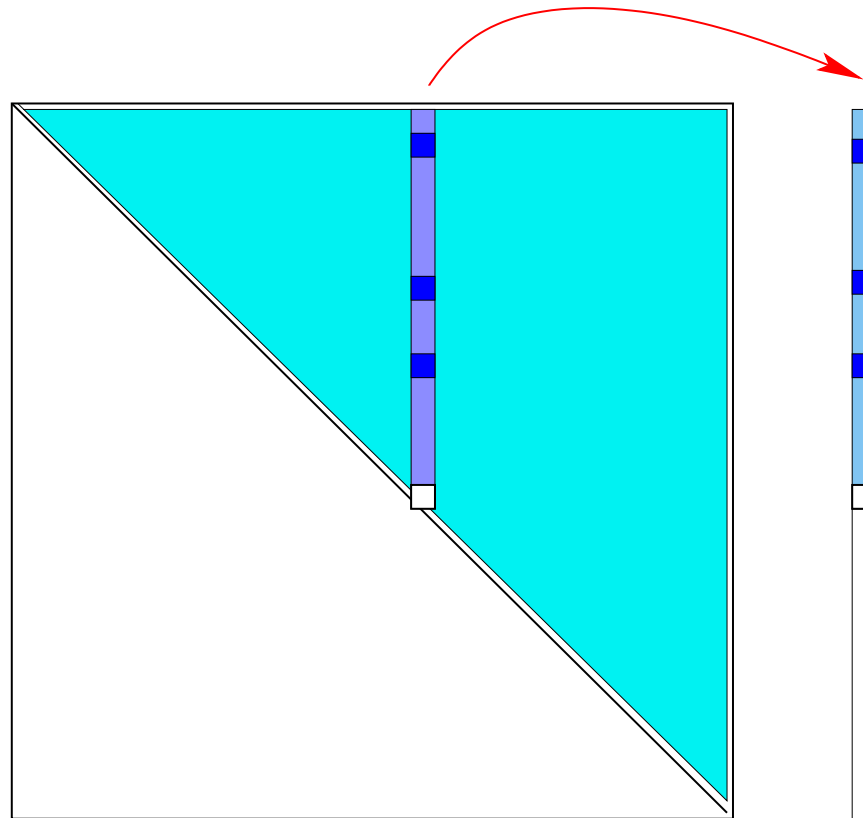
 2 Justify the above algorithm [Show that it does indeed give the solution]

Illustration for sparse case (Sparse A , dense b)



- Uses the CSC format – (CsMat struct for columns of A)
- Sparse BLAS : sparse ‘saxpy’

- Assumes diagonal entry stored first in inverted form


```
void UsolC(csptr mata, double *b, double *x)
{
    int i, k, *ki;
    double *ma;
    for (i=mata->n-1; i>=0; i--) {
        ja = U->ja[i];
        ma = U->ma[i];
        x[i] *= ma[0];
        // Note: diag. entry avoided
        for( j = 1; j < U->nzcount[i]; j++ )
            x[ja[j]] -= ma[j] * x[i];
    }
}
```


 3 Operation count ?

Sparse A and sparse b

Illustration: Consider solving $Lx = b$ in the situation:

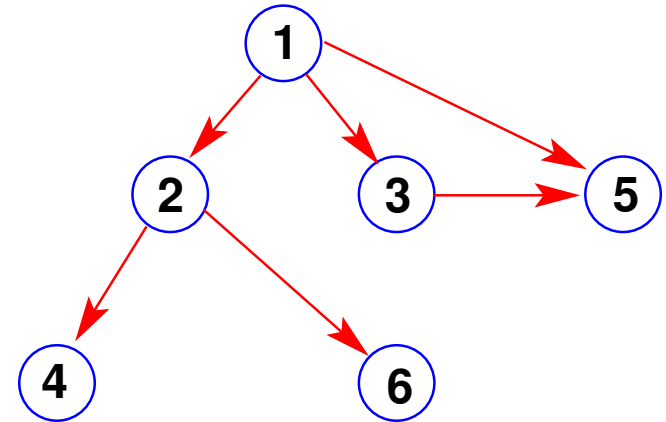
$$L = \begin{array}{|c|c|c|c|c|c|} \hline * & & & & & \\ \hline * & * & & & & \\ \hline * & & * & & & \\ \hline & * & & * & & \\ \hline * & & * & & * & \\ \hline & * & & & & * \\ \hline \end{array} \quad b = \begin{array}{|c|} \hline * \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

 4 Show progress of the pattern of $x = L^{-1}b$ by performing symbolically a column solve for system $Lx = b$.


 5 Show how this pattern can be determined with Topological sorting. Generalize to any sparse b .

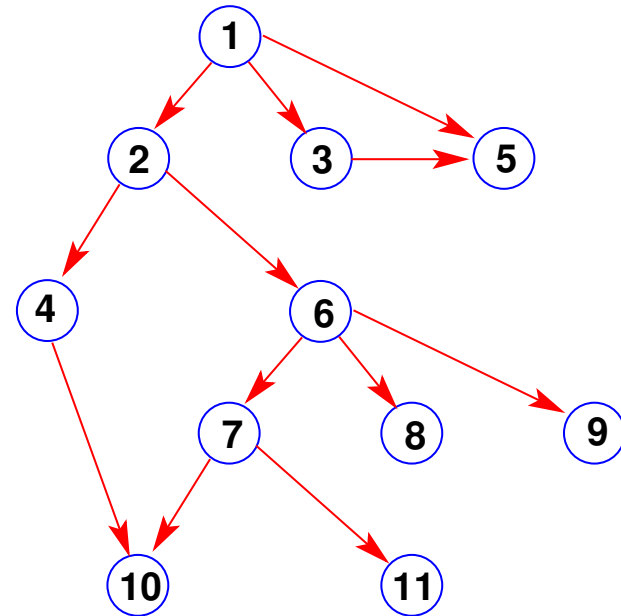
Sparse A and sparse b: Example

- Triangular system of previous example
- DAG shown in next figure
- Sets dependencies between tasks:
- Edge $i \rightarrow j$ means $a(j, i) = 1$ (j requires i)
- Root: node 1 (see right-hand side b)
- Topological sort: 1, 3, 5, 2, 6, 4 [as produced by a DFS from 1]
- In many cases, this leads to a short traversal



 Example: remove link $1 \rightarrow 2$ and redo

7 Consider a triangular system with the following graph where b has nonzero entries in positions 3 and 7. (1) Progress of solution based on Topolog. sort; (2) Pattern of solution. (3) Verify pattern with matlab.



8 Same questions if b has (only) a nonzero entry in position 1.

LU factorization from sparse triangular solves

- LU factorization built one column at a time. At step k :

We want:
$$\underbrace{L_k}_{n \times n} \underbrace{U_k}_{n \times k} = \underbrace{A_k}_{n \times k} \quad (\equiv A(1:n, 1:k))$$

$$\left[\begin{array}{ccc|ccc} 1 & & & & & \\ * & 1 & & & & \\ * & * & 1 & & & \\ * & * & * & 1 & & \\ * & * & * & ? & 1 & \\ * & * & * & ? & & 1 \\ * & * & * & ? & & \end{array} \right] \left[\begin{array}{ccc|c} x & x & x & ? \\ & x & x & ? \\ & & x & \cdot \\ & & & ? \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{array} \right] = A_k$$

- In blue: has been determined. In red: to be determined

- Step 0: Set the terms ? in L_k to zero. Result $\equiv \tilde{L}_k$
- Step 1 : Solve $\tilde{L}_k w = a_k$ [Sparse \tilde{L}_k , sparse RHS]
- Step 2: set

$$u = \begin{array}{|c} w_1 \\ w_2 \\ \vdots \\ w_k \\ \hline 0 \\ \vdots \\ 0 \\ 0 \end{array} \quad z = \frac{1}{w_k} \begin{array}{|c} 0 \\ \vdots \\ 0 \\ \hline 0 \\ w_{k+1} \\ w_{k+2} \\ \vdots \\ w_n \end{array}$$

- Then $L_k U_k = A_k$ with

$$\underbrace{\begin{bmatrix} 1 & & & & & & \\ * & 1 & & & & & \\ * & * & 1 & & & & \\ * & * & * & 1 & & & \\ * & * & * & z_{k+1} & 1 & & \\ * & * & * & \vdots & & 1 & \\ * & * & * & z_n & & & 1 \end{bmatrix}}_{L_k} ; \quad \underbrace{\begin{bmatrix} x & x & x & u_1 \\ & x & x & u_2 \\ & & x & \vdots \\ & & & u_k \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{bmatrix}}_{U_k}$$

- Verification: Note $L_k = \tilde{L}_k + ze_k^T$; Also $\tilde{L}_k z = z$
- Must verify only $L_k U_k(:, k) = a_k$, i.e., $L_k u = a_k$

$$\begin{aligned}
 L_k u &= (\tilde{L}_k + ze_k^T)u = \tilde{L}_k(I + ze_k^T)u \\
 &= \tilde{L}_k(u + w_k z) = \tilde{L}_k w = a_k
 \end{aligned}$$

- Key step: solve triangular system
- In sparse case: sparse triangular system with sparse right-hand side
- Use topological sorting at each step
- Scheme derived from this known as ‘left-looking’ sparse LU –
- Also known as ‘Gilbert and Peierls’ approach
- Reference: **J. R. Gilbert and T. Peierls**, Sparse partial pivoting in time proportional to arithmetic operations, *SIAM J. Sci. Statist. Comput.*, 9 (1988), pp. 862-874

 Benefit of this approach: Partial pivoting is easy. Show how you would do it.