Math 8602: REAL ANALYSIS. Spring 2016.
Problems for Final Exam on Tuesday, May 10, 10:00 am–Noon, VinH 211.
Office hours before Exam: Monday, May 9, 10:00 am–Noon, VinH 231.

This Final Exam will be based on the material covered by the previous homework assignments and Midterm exams, and also elements of L^p spaces and Fourier transforms, which were discussed in class (and mostly contained in Sec. 6.1–6.2, 8.1–8.3.) You will have 2 hours (120 min) to work on 6 problems, 4 of which will be selected from the following list.

No books. No electronic devices. You can use class notes.

#1. Let f be a Lebesque measurable function on $\mathbb{R}^1$ such that

$$f(x+y) = f(x) + f(y) \quad \text{for all} \quad x, y \in \mathbb{R}^1.$$

Show that $f(x) = cx$ for some constant c.

#2. Let f, g be functions in the linear space $L^p(X, \mathcal{M}, \mu)$, $0 < p < \infty$, with quasinorm $||\cdot||_p$, which are defined on p. 181. Show that

$$||f+g||_p \le K(p)\cdot\big(||f||_p + ||g||_p\big), \quad \text{where} \quad K(p) \quad \text{is a constant such that} \quad K(p) \searrow 1 \quad \text{as} \quad p \nearrow 1.$$

Hint. For $0 < p < 1$, we have $|f+g|^p \le |f|^p + |g|^p$.

#3. (Strict convexity of L^p). In the case $1 < p < \infty$, show that from $||f_1||_p = ||f_2||_p = 1$, $f_1 \ne f_2$, and $f := \theta f_1 + (1-\theta) f_2$ with $0 < \theta < 1$, it follows a **strict** inequality $||f||_p < 1$.

#4. For $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ with $p \ge 1$, show that the convolution $f * g$ (defined on p. 239) belongs to $L^p(\mathbb{R}^n)$ and satisfies $||f * g||_p \le ||f||_1 \cdot ||g||_p$.

#5 (see also Exercise 14 on p. 254–255). Let A be the set of all smooth functions u on $\mathbb{R}^1$, satisfying

$$u(x+2\pi) \equiv u(x), \qquad \int_0^{2\pi} u\,dx = 0, \qquad \int_0^{2\pi} u^2\,dx \ge 1.$$

Find

$$\inf_{u \in A} \int_0^{2\pi} \left(\frac{du}{dx}\right)^2 dx.$$

Hint. Use Fourier expansion of u.

#6. Consider the family of functions on $\mathbb{R}^1$:

$$K_\varepsilon(x) := \frac{\varepsilon}{\pi(x^2+\varepsilon^2)}, \quad \varepsilon > 0.$$

Show that the convolution

$$K_{\varepsilon_1} * K_{\varepsilon_2} \equiv K_{\varepsilon_1+\varepsilon_2} \quad \text{for} \quad \varepsilon_1, \varepsilon_2 > 0.$$