Math 8602: REAL ANALYSIS. **Spring 2016**

Homework #3 (due on Wednesday, March 9).
40 points are divided between 4 problems, 10 points each.

#1. Show that the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(4^k x)}{2^k}$$

is continuous on $\mathbb{R}^1$, but its variation $V[f;\, a, b] = \infty$ for any $a < b$.

Hint. For any interval I,

$$V[f_n,\, I] \ge \int_I \cos(4^m x)\, df_n(x), \quad \text{where} \quad f_n(x) := \sum_{k=n}^{\infty} \frac{\sin(4^k x)}{2^k}.$$

#2. (Problem 36 on p. 127). Let X be the set of all real-valued Lebesgue measurable functions f on $[0,1]$ satisfying the inequality $|f| \le 1$. Show that there is NO topology $\mathcal{T}$ on X such that $f_n \to 0$ a.e. as $n \to \infty$ if and only if it converges with respect to $\mathcal{T}$.

#3. Let f be a real valued continuous function on $\mathbb{R}^1$ such that $f(x) \equiv 0$ for $|x| \ge 2$. Show that

$$f^{(\varepsilon)}(x) := \int_{\mathbb{R}^1} f(x - \varepsilon y)\varphi(y)dy \;\to\; f(x) \quad \text{as} \quad \varepsilon \searrow 0$$

uniformly on $\mathbb{R}^1$, where

$$\varphi(y) := \frac{1}{\sqrt{\pi}} \cdot e^{-y^2}.$$

#4. Use the previous problem for the proof of the Weierstrass theorem: every continuous function on $[-1,1]$ can be uniformly approximated by polynomials.