Math 8602: REAL ANALYSIS. Spring 2016

Homework #4. Problems and Solutions.

#1. Let $\mathcal{K}$ be a family of all nonempty closed subsets of $[0,1]\times[0,1]$ with respect to the Euclidean distance. Show that $\mathcal{K}$ is a metric space with the Hausdorff distance

$$\rho(A,B) := \max\Big\{\max_{x\in A}\operatorname{dist}(x,B),\ \max_{y\in B}\operatorname{dist}(y,A)\Big\},\quad \operatorname{dist}(x,B) := \min_{y\in B}|x-y|,\quad \text{etc.}$$

Proof. We have to verify the axioms of a metric space:

$$(i)\ \rho(A,B)=0\ \Rightarrow\ A=B,\quad (ii)\ \rho(A,B)=\rho(B,A),\quad \text{and}\quad (iii)\ \rho(A,C)\le\rho(A,B)+\rho(B,C).$$

The equality $\rho(A,B)=0$ for nonempty closed subsets A and B simply means that $A\subseteq B\subseteq A$, i.e. $A=B$, so that we have (i). The property (ii) is obvious. For the proof of (iii), note that

$$r\ge\max_{x\in A}\operatorname{dist}(x,B)\quad\Longleftrightarrow\quad A\subseteq B^r := \{x:\ \operatorname{dist}(x,B)\le r\}.$$

Therefore,

$$\rho(A,B)=\min\Big\{r\ge 0:\quad A\subseteq B^r,\quad B\subseteq A^r\Big\}. \tag{1}$$

Set $r_1:=\rho(A,B)$, $r_2:=\rho(B,C)$. Then

$$B\subseteq C^{r_2},\quad A\subseteq B^{r_1}\subseteq (C^{r_2})^{r_1}=C^{r_1+r_2}.$$

By symmetry, we also have $C\subseteq A^{r_1+r_2}$. This implies (iii): $\rho(A,C)\le r_1+r_2=\rho(A,B)+\rho(B,C)$.

#2. Show that in the previous problem, the metric space $(\mathcal{K},\rho)$ is compact.

Proof. It suffices to verify that the metric space $(\mathcal{K},\rho)$ is (i) totally bounded and (ii) complete.

(i). Fix $\varepsilon>0$ and take an arbitrary finite family of closed sets $F_1,\dots,F_m$, such that

$$[0,1]\times[0,1]\subseteq\bigcup_{j=1}^m F_j,\quad\text{and}\quad\max_j\operatorname{diam}F_j\le\varepsilon.$$

Then the family $S:=\sigma(\{F_j\})$ consists of all possible unions of subfamilies of $\{F_j\}$, including the empty set. The family S consists of at most 2^m elements. For an arbitrary $A\in\mathcal{K}$, take

$$B:=\bigcup\{F_j:\quad F_j\cap A\quad\text{is nonempty}\}\ \in\mathcal{K}.$$

Then $A\subseteq B\subseteq A^\varepsilon:=\{x:\ \operatorname{dist}(x,A)\le\varepsilon\}$. By (1), this means that $\rho(A,B)\le\varepsilon$. In other words, $\mathcal{K}$ is totally bounded:

$$\min_{B\in S}\rho(A,B)\le\varepsilon,\quad\forall A\in\mathcal{K}.$$

(ii) Let $\{A_j\}$ be a Cauchy sequence in $(\mathcal{K},\rho)$. We can assume that $\rho(A_j,A_{j+1})\le\varepsilon_j:=2^{-j}$ for all $j=1,2,\dots,n$, because otherwise we can take a subsequence $\{A_{k_j}\}$ instead of $\{A_j\}$. Introduce

$$B_j:=A_j^{2\varepsilon_j}:=\{x:\ \operatorname{dist}(x,A_j)\le 2\varepsilon_j\}\in\mathcal{K}.$$

Then by (1),

$$A_{j+1}\subseteq A_j^{\varepsilon_j},\quad\text{and}\quad B_{j+1}\subseteq A_{j+1}^{2\varepsilon_{j+1}}=A_{j+1}^{\varepsilon_j}\subseteq(A_j^{\varepsilon_j})^{\varepsilon_j}=A_j^{2\varepsilon_j}=B_j.$$

Hence

$$B_j\searrow B:=\bigcap_{j=1}^\infty B_j\in\mathcal{K}\quad\text{as}\quad j\to\infty.$$

On the other hand, we have the following

Exercise. $\forall \varepsilon > 0,\ \exists N$ such that $B_j \subseteq B^\varepsilon,\ \forall j \geq N$.

In combination with $B \subseteq B_j := A_j^{2\varepsilon_j}$, this implies

$$\rho(A_j, B) \leq \max\{\varepsilon, 2\varepsilon_j\}, \quad \forall j \geq N; \quad \text{and} \quad \limsup_{j\to\infty} \rho(A_j, B) \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get the desired convergence in $(\mathcal{K}, \rho)$: $\rho(A_j, B) \to 0$ as $j \to \infty$.

#3. Let $K(x, y)$ be a continuous function on $[0, 1] \times [0, 1]$. Consider the metric space $\big(C([0, 1]), \rho\big)$, where

$$\rho(f, g) := \max_{[0,1]} |f - g|.$$

Show that the family of functions

$$A := \left\{ F(x) := \int_0^1 K(x, y)\, f(y)\, dy : \quad f \in C([0, 1]), \quad \max_{[0,1]} |f| \leq 1 \right\}$$

is a precompact subset of $\big(C([0, 1]), \rho\big)$. Verify whether or not it is compact.

Proof. It is known that K is bounded and uniformly continuous on $Q := [0, 1] \times [0, 1]$, i.e.

$$\sup_Q |K| \leq M = \text{const} < \infty, \quad \text{and} \quad \omega(\rho) := \sup_{|z_1 - z_2| \leq \rho} |K(z_1) - K(z_2)| \to 0 \quad \text{as} \quad \rho \searrow 0.$$

These properties are obviously preserved for functions $F \in A$:

$$\sup_{[0,1]} |F| \leq M, \quad \text{and} \quad \sup_{|x_1 - x_2| \leq \rho} |F(x_1) - F(x_2)| \leq \omega(\rho).$$

This means that the family A is uniformly bounded and equicontinuous. By Theorem 4.4.3, it is precompact, i.e. its closure in $\big(C([0, 1]), \rho\big)$ is compact.

The family A is not necessarily compact. Indeed, consider $K(x, y) := (x - y)^+ = \max(x - y, 0) \in C$. Then for every $f \in C$, the corresponding function

$$F(x) = \int_0^x (x - y) f(y)\, dy, \quad F'(x) = \int_0^x f(y)\, dy, \quad F''(x) = f(x) \in C.$$

In other words, $F \in C^2$. Now let g_n be a sequence of functions in C which converges in L^1 to a discontinuous function $g \equiv 0$ on $[0, 1/2]$, $g \equiv 1$ on $(1/2, 1]$. The corresponding functions G_n converge to G in C (even in C^1), but $G'' = g \notin C$, $G \notin C^2$. Then, G cannot belong to A, and A is not complete and therefore not compact.

#4 (§4.3, p. 147.) Let $(\mathcal{F}, \lesssim)$ be a filter directed under reverse inclusion, i.e.

$$F_1 \lesssim F_2 \quad \Longleftrightarrow \quad F_2 \subseteq F_1.$$

A net $< x_F >_{F\in\mathcal{F}}$ is **associated** to $\mathcal{F}$ if $x_F \in F$ for every $F \in \mathcal{F}$. Show that

$$\mathcal{F} \to x \quad \Longleftrightarrow \quad \text{every associated net} \quad < x_F >_{F\in\mathcal{F}} \to x.$$

Proof. First suppose $\mathcal{F} \to x$. This means that if x belongs to an open set G, then $G \in \mathcal{F}$. If a net $< x_F >_{F\in\mathcal{F}}$ is associated to $\mathcal{F}$, then $\forall F \gtrsim G$, we have $x_F \in F \subseteq G$. By definition, $< x_F >_{F\in\mathcal{F}} \to x$.

Now suppose $\mathcal{F}$ does not converge to x. Then $\exists$ an open set $G \notin \mathcal{F}$ such that $x \in G$. Note that $\forall F \in \mathcal{F}$, the inclusion $F \subseteq G$ is impossible (by definition of a filter, this would imply $G \in \mathcal{F}$). Therefore, $\forall F \in \mathcal{F}$, $\exists x_F \in F \backslash G$, which means that x_F does not converge to x.