Maximal Independent Set and Minimum Connected Dominating Set in Unit Disk Graphs

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December 13, 2004
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Abstract

In ad hoc wireless networks, the connected dominating set can be used as a virtual backbone to improve the performance. Many constructions for approximating the minimum connected dominating set are based on construction of maximal independent set. The relation between the size $mis(G)$ of a maximum independent set and the size $cds(G)$ of minimum connected dominating set in the same graph $G$ plays an important role in establishing the performance ratio of those approximation algorithms. Previously, it is known that $mis(G) \leq 4 \cdot cds(G) + 1$ for all unit disk graph $G$. In this paper, we improve it by showing $mis(G) \leq 3.8 \cdot cds(G) + 1.2$.

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1 Introduction

Wireless Sensor Network has been widely used in many industries such as the healthcare industry, food industry, and agriculture [11]. It consists of many sensors each of which is not only a mobile host but also a router. In other words, the sensors are able to forward the received data packages according to routing protocols. Usually, the sensors are cheap devices with the identical design. In this situation, every sensor has the same power and hence can communicate with others within a unit distance so that, the topology of the sensor network can be formulated as a unit disk graph.

A unit disk is a disk with radius one. A unit disk graph is associated with a set of unit disks in the Euclidean plane. Each node is the center of a unit disk. An edge exists between two nodes $u$ and $v$ if and only if $|uv| \leq 1$ where $|uv|$ is the Euclidean distance between $u$ and $v$. This means that two nodes connecting with an edge if and only if $u$'s disk covers $v$ and $v$'s disk covers $u$.

A subset of vertices in a graph is called a dominating set if every vertex is either in the subset or adjacent to a vertex in the subset. A dominating set is connected if it induces a connected subgraph. The connected dominating set is often used as a virtual backbone in wireless sensor networks to improve communication and storage performance [6]. Clearly, the smaller virtual backbone gives the better performance. However, computing the minimum connected dominating set is NP-hard even in unit disk graphs. Therefore, many efforts [2, 12, 13, 15, 14, 1] have been made to design approximations or heuristics for the minimum connected dominating set. Among them, several constructions [14, 4, 9] work in a popular way: First, construct a dominating set and then connect it by adding more vertices.

Since every maximal independent set is a dominating set and it is easy to construct, one usually constructs a maximal independent set at the first step. Therefore, the approximation performance ratio would be determined by two facts. The first is how large a maximal independent set compared with a minimum connected dominating set. The second is how many vertices required to connect a maximal independent set. It is showed in [14] that in
every unit disk graph $G$,
\[ mis(G) \leq 4 \cdot cds(G) + 1 \]
where $mis(G)$ is the size of a maximum independent set and $cds(G)$ is the size of a minimum connected dominating set in $G$. In this paper, we show that
\[ mis(G) \leq 3.8 \cdot cds(G) + 1.2. \]
Therefore, all evaluation of performance ratios in [14, 4, 9] are improved.

2 Preliminary

We call a unit disk (including its boundary) at center $x$ the \textit{neighbor area} of $x$, denoted by $N(x)$. In general, for a graph $G$, its \textit{neighbor area} $N(G)$ is defined to be the union of all neighbor areas of its vertices. Throughout this paper, the study is referred to a unit disk graph $G$. Therefore, two vertices $u$ and $v$ are said to be \textit{adjacent} if $|uv| \leq 1$ and \textit{independent} if $|uv| > 1$.

The following lemma can be found in [14].

\textbf{Lemma 1} \textit{The neighbor area of a vertex contains at most five independent vertices.}

This lemma is proved by noting the following fact: Suppose $u$ and $v$ are two independent vertices in the neighbor area $N(x)$ of a vertex $x$. Then we must have $\angle uxv > 60^\circ$. In fact, if $\angle uxv \leq 60^\circ$, then $|uv| \leq \max(|ux|, |vx|) \leq 1$, contradicting the independence of $u$ and $v$. This elementary fact will be used later without mentioning.

For simplicity of speaking, we say that points $x_1, x_2, \ldots, x_k$ \textit{counter-clockwise}ly lying in $N(x)$, if $xx_1, xx_2, \ldots, xx_k$ lie in counter-clockwise direction around $x$. When $x_1, x_2, \ldots, x_k$ are independent, we have $\angle x_1xx_2 > 60^\circ$, $\ldots$, $\angle x_{k-1}xx_k > 60^\circ$ and $\angle x_kxx_1 > 60^\circ$ and hence $k60^\circ < 360^\circ$. This implies $k < 6$, that is $k \leq 5$.

Now, consider three vertices $a, b, c$ of a regular triangle with unit edge length. Connect two vertices $a$ and $b$ by an arc of unit radius at center $c$, connect two vertices $b$ and $c$ by an arc of unit radius at center $a$, and connect two vertices $a$ and $c$ by an arc of unit radius at center $b$. Let $A$ be the area surrounded by the three arcs (Fig. 1). We call $A$ a \textit{unit}
arc-triangle abc. It is a well-known fact that every two points in the area $A$ have distance at most one. Therefore, we have

**Lemma 2** The unit arc-triangle $A$ cannot contain two independent vertices.

### 3 Main Results

By Lemma 1, in the neighbor area of two adjacent vertices, there are at most nine independent vertices. However, the following lemma gives a better result.

**Lemma 3** The neighbor area of two adjacent vertices contains at most eight independent vertices.

**Proof.** Consider two adjacent vertices $u$ and $v$. For contradiction, suppose that the neighbor area of $u$ and $v$ contains an independent set $I$ of more than eight vertices. First, we claim that the intersection $A = N(u) \cap N(v)$ contains exactly one vertex in $I$. In fact, if $A$ contains $k$ vertices in $I$, then by Lemma 1, $N(u) - A$ contains at most $5 - k$ vertices in $I$ and $N(v) - A$ contains at most $5 - k$ vertices in $I$. Therefore, $N(u, v)$ contains at most $10 - k$ vertices in $I$. Hence, $10 - k \geq 9$, that is, $k \leq 1$. Let $x$ and $y$ be two intersection points of boundaries of $N(u)$ and $N(v)$. Since $|uv| \leq 1$, we have $\angle xuv = \angle yux \geq 120^\circ$. Thus, $N(u) - A$ contains at most four vertices in $I$, so does $N(v) - A$. This means that $I$ contains at most $8 + k$ vertices and hence $8 + k \geq 9$, that is, $k \geq 1$.

Let $a_0$ be the unique vertex in $I$, lying in $N(u) \cap N(v)$. As a consequence, each of $N(u) - A$ and $N(v) - A$ contains exactly four vertices in $I$ and $|I| = 9$. Suppose $I = \{a_0, a_1, \ldots, a_8\}$,
$a_0, a_1, \ldots, a_4$ lie counter-clockwisely in $N(u)$ and $a_0, a_5, \ldots, a_8$ lie counter-clockwisely in $N(v)$. Denote by $u_b_i$ the radius containing $a_i$ for $i = 2, \ldots, 4$ and by $v_b_i$ the radius containing $a_i$ for $i = 5, \ldots, 8$. Draw four unit arc-triangles $u_b_2c_2, u_b_3c_3, v_b_6c_6$, and $v_b_7c_7$ as shown in Fig. 2. Their boundaries intersect the boundary of $N(u) \cap N(v)$ at $d_2, d_3, d_6, d_7$, respectively. Note that none of $a_1, a_4, a_5, a_8$ can lie in the four unit arc-triangles and $N(u) \cap N(v)$. Therefore, $a_1, a_4, a_5, a_8$ must lie in the four small dark areas $xc_2d_2, yc_3d_3, yc_6d_6$ and $xc_7d_7$, respectively, as shown in Fig. 2.

![Figure 2: Four small dark areas.](image)

Next, we will show that there exist two small dark areas too close to contain two independent vertices, a contradiction. To do so, we note that $\angle b_2u_b_3 > 60^\circ$ and $\angle c_2u_b_2 = \angle b_3u_c_3 = 60^\circ$. Hence, $\angle c_2u_c_3 > 180^\circ$ and $\angle c_3u_c_2 < 180^\circ$ (here, please note that $\angle c_3u_c_2$ is the one obtained by moving $c_3u$ counterclockwisely to $c_2u$). Similarly, $\angle c_7u_a_6 < 180^\circ$. Therefore $\angle u_c_2c_7 + \angle c_2c_7v + \angle d_6c_3 + \angle c_6d_3u > 360^\circ$. Hence, either $\angle u_c_2c_7 + \angle c_2c_7v > 180^\circ$ or $\angle d_6c_3 + \angle c_6d_3u > 180^\circ$. Assume the former occurs without loss of generality (Fig. 3). We show that dark areas $xc_2d_2$ and $xc_7d_7$ cannot contain two vertices in $I$. 

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Figure 3: Turn unit arc-triangle $vb_7c_7$ until $vc_7 \parallel uc_2$.

To do so, we first enlarge area $xc_7d_7$ by turning the unit arc-triangle $vb_7c_7$ around $v$ until $vc_7$ is parallel to $uc_2$. At this limit position, quadrilateral $c_2uvc_7$ becomes a parallelogram so that $|c_2c_7| = |uv| \leq 1$. It is easy to see that the distance between two points in areas $xc_2d_2$ and $xc_7d_7$ cannot exceed $\max(|c_2c_7|, |c_2d_7|, |d_2c_7|, |d_2d_7|)$. Moreover, we claim that $|d_2d_7| \leq \max(|c_2d_7|, |d_2c_7|)$. In fact, note that $\angle c_7d_7d_2 + \angle d_7d_2c_2 > 180^\circ$. Thus, either $\angle c_7d_7d_2 > 90^\circ$ or $\angle d_7d_2c_2 > 90^\circ$. Therefore, either $|d_2c_7| > |d_2d_7|$ or $c_2d_7 > |d_2d_7|$, that is, our claim is true.

Now, to complete the proof of the lemma, it remains to prove that $|c_2d_7| \leq 1$ and $|d_2c_7| \leq 1$.

To see $|c_2d_7| \leq 1$, we first make $|c_2d_7|$ longer by moving $v$ away from $u$ until $|uv| = 1$ (Fig. 4). At this limit position, we have $|uv| = |vb_7| = |b_7d_7| = |d_7u| = 1$. Therefore, $uvb_7d_7$ is a parallelogram. Hence, $|d_7b_7| = |c_2c_7| = 1$ and $d_7b_7$ is parallel to $uv$ and hence parallel to $c_2c_7$. It follows that $c_2d_7b_7c_7$ is a parallelogram. Thus, $|c_2d_7| = |b_7c_7| = 1$. Similarly, we can show $|d_2c_7| \leq 1$.

The following two lemmas are about properties of graphs.
Figure 4: Move $u$ until $|uw| = 1$.

**Lemma 4** For any unit disk graph, there exists a minimum spanning tree such that every vertex has degree at most five.

*Proof.* Let $T$ be a minimum spanning tree. It is easy to see that $T$ must have the following two properties:

(a1) Two edges meeting at a vertex form an angle of at least $60^\circ$.

(a2) If two edges form an angle of exactly $60^\circ$, then they have the same length.

It follows immediately from (a1) that every vertex in $T$ has degree at most six. Consider a vertex $u$ with degree six in $T$. By (a1), every angle at $u$ equals $60^\circ$. By (a2), all edges incident to $u$ have the equal length. These two facts imply that every vertex $v$ adjacent to a vertex $u$ of degree six has degree at most four. In fact, $u$ has two edges $uw$ and $ux$ such that $\angle uvw = \angle uwx = 60^\circ$ and $|uw| = |uw| = |ux|$. It follows that $|vw| = |uw|$ and $|vx| = |uw|$. Thus, replacing $uw$ and $ux$ by $vw$ and $vx$ results in still a minimum spanning tree. But, $v$ gets two more edges in the new tree. Hence, $v$ has degree at most four in the original tree. Now, if we do only one replacement, that is, replace $uw$ by $vw$, but do not replace $ux$ by $vx$. Then both $u$ and $v$ have degree at most five.
One may worry about that \( v \) may be adjacent to more than one vertices of degree six. In such a case, could \( v \) receives too many edges and its degree increases to six? This cannot happen because after all replacements, resulting tree is still a minimum spanning tree and hence it still has property that every vertex adjacent to a vertex of degree six has degree at most four. However, \( v \) would be adjacent to at least one vertex of degree five in the new tree. It follows that \( v \) cannot have degree six.

\[ \Box \]

**Lemma 5** Every tree \( T \) with at least three vertices has a non-leaf vertex adjacent to at most one non-leaf vertex.

**Proof.** Let \( T' \) be the subtree obtained from \( T \) by removal of all leave. Since \( T \) has at least three vertices, \( T' \) contains at least one vertex. If \( T' \) contains only one vertex, then it meets our requirement. If \( T' \) contains more than one vertices, then every leaf of \( T' \) is a non-leaf vertex of \( T \) satisfying the condition stated in the lemma.

\[ \Box \]

Now, we are ready to show our main theorem.

**Theorem 1** For any unit disk graph \( G \), the size of a maximal independent set is at most \( 3.8\text{clos}(G) + 1.2 \) where \( \text{clos}(G) \) is the size of a minimum connected dominating set.

**Proof.** Let \( G \) be a subgraph induced by a minimum connected dominating set in the given unit disk graph. Then \( G \) is a unit disk subgraph. By Lemma 4, \( G \) has a minimum spanning tree \( T \) such that every vertex has degree at most five. Let \( |T| \) denote the number of vertices in \( T \). We will show by induction on \( |T| \) that there exists at most \( 3.8|T| + 1.2 \) independent vertices in the neighbor area of \( T \). For \( |T| = 1 \) or 2, this is true by Lemmas 1 and 3. Next, we assume \( |T| \geq 3 \). By Lemma 5, \( T \) contains a non-leaf vertex \( v \) adjacent to at most one non-leaf vertex. Let \( u \) be the non-leaf neighbor of \( v \) if it exists, or a leaf neighbor of \( v \), otherwise. Let \( x_1, \ldots, x_k \) \((k \leq 4)\) be other neighbors of \( v \). Note that for each \( x_i \) for \( 1 \leq i \leq k-1 \) its neighbor area contains at most four independent vertices also independent from \( v \) by Lemma 1 and the neighbor area of \( v \) and \( x_k \) contains at most seven independent vertices also independent from \( u \) by Lemma 3. Moreover, by the induction hypothesis, the
neighbor area of $T - v, x_1, \ldots, x_k$ contains at most $3.8(|T| - k - 1) + 1.2$ independent vertices. Therefore, the neighbor area of $T$ contains at most

$$3.8(|T| - k - 1) + 1.2 + 7 + 4(k - 1) = 3.8|T| + 1.2 + 0.2(k - 4) \leq 3.8|T| + 1.2$$

independent vertices. Note that $|T| = c_{ds}(G)$. This completes the proof of the theorem. \qed

As a corollary, we have

**Corollary 1.** For approximation algorithms in $[14, 4]$ for the minimum connected dominating set, the performance ratio can be reduced from 8 to 7.8.

### 4 Discussion

A 4-star is a graph with a center and four leaves. We believe conjecture that

**Conjecture 1.** The neighbor area of a 4-star subgraph in a unit disk graph contains at most twenty independent vertices.

If this true, then by a similar argument, Theorem 1 can be improved from 3.8 to 3.6. However, dealing with twenty points with elementary geometric method is quite hard. Therefore, the proof of this conjecture may need some advanced methods for packing and covering, such as harmonic analysis. In fact, this conjecture can be easily transformed to a unit disk packing problem if we double the radius for those disks in construction of the neighbor area.

A weakly connected dominating set is a dominating set such that putting edges between dominers and edges between dominers and dominines results in a connected graph [3, 7]. The weakly connected dominating set has also been used in wireless networks. In [7], it is showed that some special constructed maximal independent sets can be weakly connected. By Lemma 1, those maximal independent sets are approximation solutions within a factor of 5 from the minimum weakly connected dominating set. Could we improve this factor with Lemma 3? It is hard to answer since it is possible, but, not easy to obtain such an
improvement. The difficulty is that two dominers may be connected through a dominee which is not in considered maximal independent set.

Acknowledgement: Authors wish to thank Dr. Manki Min and Dr. Xiuzhen Cheng for pointing out the possibility that the neighbor area of two adjacent vertices may contain at most eight independent vertices. Our work is motivated from this possibility.

References


