Technical Report

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Bearing-Only Active Target Localization Strategies for a System of Two Communicating Mobile Robots: Full Technical Report

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***Abstract*— We study the problem of locating a stationary target using two robots equipped with bearing sensors. The goal is to reduce the uncertainty in the target's location to a value below a given threshold in minimum time. Our cost formulation explicitly models time spent in traveling as well as taking measurements. Further, the robots are subject to distance-based communication constraints. We first study properties of an optimal strategy which has access to the target's true location. It turns out that under certain circumstances, the optimal algorithm will break communication to take measurements and rendezvous to merge them.**

Using these insights, we design an online strategy (which does not have access to the target's true location) and show that the strategy can locate the target up to a desired uncertainty level at near-optimal cost. The results are applicable to other bearing sensors with non-zero measurement cost, such as pan-tilt cameras. In addition to theoretical analysis, we validate the algorithm in simulations and field experiments performed using autonomous surface vehicles.

Fig. 1. The robotic systems used in field experiments. The 2-meter long boats were designed to track invasive fish autonomously. Each is equipped with wireless communications, directional antenna used as bearing sensors, a navigation suite, and computing hardware. Our algorithm was implemented on this system which was shown to localize targets with small uncertainty.

I. Introduction

Systems of networked, mobile sensors are expected to have significant impact on environmental monitoring applications by automating tedious and potentially dangerous sensing tasks. Once such area we have been studying is deploying robots to assist in locating radio-tagged invasive fish [1]. Our system consists of Autonomous Surface Vehicles (ASVs) which have radio antennas to detect radio tags attached to the fish (Figure 1). The system is intended to provide data on long-term motions of invasive fish and search large areas for stationary fish aggregations. The duration of such tasks forces us to consider the trade-off between time spent to localize each radio tag and system life; our active-localization algorithm must choose measurement locations which provide good information about each target, but do not produce an overly time-consuming trajectory. The issue is more complicated when two robots must synchronize their estimate of the target location, since we simultaneously optimize their communication. Therefore, in this work we study the general problem of active localization for two robots subject to distance-based communication constraints.

In this paper, we provide three main results. First, we extend the existing body of work which analyzes the offline case: planning measurements with respect to a known target location. In this area we provide the optimal two-robot algorithm, which was previously unknown. We present the algorithm details in Section IV.

We then extend the offline-optimal algorithm to include communication constraints, in Section IV-A. While our strategy guarantees that the robots will be within communication radius when they exchange measurements, we show that the optimal strategy may break communication to take measurements. Thus, enforcing communication in our setting is potentially suboptimal and our algorithm allows the robots to break communication when necessary.

Finally, we address the more realistic case: when only a prior estimate of the true target location is known. To solve this *online* problem, an algorithm must plan measurements to minimize the worst-case cost, even though the prior estimate may be uncertain or even misleading. We provide an online algorithm whose cost is at most a logarithmic factor more than the *optimal* algorithm with arbitrarily high probability. Ours is the first algorithm to provide such a guarantee. The details of our solution are discussed in Section V.

As a final contribution, we have implemented and tested the algorithm in field experiments on a mobile sensor network (see Figure 1). We present experiments demonstrating the algorithm and successfully locating a radio transmitter to within 10 meters of ground truth with only a few measurements, and without requiring significant travel time. These field experiments are presented in Section VI-B.

II. Problem Formulation

In this section we define our notation and optimization problem. Let the true target location be $x^\star$. Two mobile robots can take bearing measurements of the target location, but each measurement takes a constant amount of time, t_m and is corrupted by zero-mean Gaussian noise with variance

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σ_s^2. We denote sensor locations as s_u and s_v for the two robots, or simply s when the assignment is irrelevant.

We define the full sequence of measurements as $S_u = \{s_u^1, \cdots, s_u^n\}$ for robot u. $S = \{S_u + S_v\}$ is the *joint* set of measurements, and $|S|$ is the total number of measurements taken between both robots. Traveling between sensor locations takes time proportional to the Euclidean distance between them. Each measurement take time t_m. Thus, total time required to travel to each sensor location in the specified order, plus take a measurement at each location is given by,

$$C(S_u) = t_m \cdot |S_u| + \text{len}(S_u) \tag{1}$$

$$\text{where, } \text{len}(S_u) = \sum_{i=1}^{|S_u|} ||s_u^i - s_u^{i-1}||_2$$

A target estimate at each time step is assumed to be a two-dimensional Gaussian, with mean $\hat{x}(t)$, and covariance $\Sigma(t)$. The uncertainty in the target's estimate is given by its covariance matrix. The covariance matrix will depend on the type of estimator used. However, from the Cramer Rao Lower Bound [2], [3] we know, for any unbiased estimator, $\Sigma \succeq \mathbb{F}^{-1}(x^\star, S)$ where $\mathbb{F}$ is the Fisher Information Matrix (FIM), with the target at $x^\star$, and measurement locations given by S. Hence, we will use the inverse of the FIM to represent the uncertainty in the target's estimate for the remainder of the paper. We optimize with respect to the maximum eigenvalue of $\mathbb{F}^{-1}$. Other uncertainty measures such as trace or determinant can be easily bounded by the maximum eigenvalue.

Our objective is to find the measurement sequences for the two robots to reduce the maximum eigenvalue of the resulting covariance below a desired level. Note $\mathbb{F}$is the inverse of the uncertainty, and the eigenvalues of a matrix inverse are the inverse of the matrix eigenvalues. With respect to $\mathbb{F}$, our information constraint is:

$$\lambda_{\min}(\mathbb{F}(x^\star, S)) \geq \lambda_d \tag{2}$$

Here λ_d defines the requested precision (information) in the final estimate. For example, fish biologists often use estimates which are accurate to 5 meter resolution ($\sqrt{\lambda_{\max}\Sigma} = 5$), so we set λ_d to $1/25$ in field experiments. For brevity, we will simply refer to the left hand side as $\lambda_{\min}(S)$ whenever $x^\star$ does not change.

We model the case when the robots may not communicate unless they are within some known radius r_c. If the robots do not meet to communicate the results of their measurements, they cannot form a joint estimate of the target's position. So we enforce that the target estimate uncertainty is a function of the measurements gathered up to the last time the robots met. Our problem can now be concisely stated as

$$\min_{S_u, S_v} \max_{i=\{u,v\}} \text{len}(S_i) + t_m \cdot |S_i| \tag{3}$$

$$\text{s.t.}$$

$$\lambda_{\min}(S) \geq \lambda_d \tag{4}$$

$$\exists t \geq |S| : |s_u^t - s_v^t| \leq r_c \tag{5}$$

Note Eq. (5) simply states the robots must meet to exchange measurements at some point after the sequence is completed. For example, in the *offline* case (as in Section IV), the true target location, $x^\star$, is known, then Eq (4) depends only on the relative locations of the sensors with respect to $x^\star$. Then the robots do not need to meet more than once, since exchanging measurements will not affect $x^\star$. However, in the online case (as in Section V), $x^\star$ is unknown. Therefore, the robots must either take enough measurements to ensure that *for any* $x^\star$, Eq. (4) is satisfied, or periodically meet, update their estimate of $x^\star$, and adjust the measurement sequence to make use of the shared information.

We can now examine the literature for related problems.

III. Related Work

Recently, there has been significant interest in designing estimators to achieve the Cramer Rao Lower Bound (CRLB). Our focus in this paper is on the complementary aspect of choosing measurement locations, and our algorithm works with any estimator which approaches the CRLB in the limit.

Most active tracking algorithms can be classified as locally optimal, such as gradient ascent (e.g., work by Grocholsky et al. [4], [5]) or track enumeration (e.g., Frew et al. [6], [7]). Zhou et al. [8] considered constraints on the target motion to find a gradient ascent to minimize the trace of the target's covariance matrix. Similarly, the work in [7] searches over next-action space for a feasible sensor trajectory. These works cannot bound the cost of the resulting trajectories.

A novel aspect of our formulation is that we optimize the trajectory of the robots with respect to a weighted sum of the measurement count and the distance traveled. We consider measurement time for a variety of reasons. In our previous work, we have used sensors which sample radio signal strength over one to two minutes to discern the bearing to the target [9]–[11]. In some works, local maneuvers are used to increase the accuracy of the sensor without significantly changing the target-robot state. For example, [12] et al. use local maneuvers to construct bearing measurements to targets. Similarly, [13] et al. use an S-shaped maneuver to resolve the direction to a target when using a hydrophone array. These maneuvers do not significantly change the robot-target configuration, but cost time and energy. Traditional active localization literature often ignores this cost.

The study of optimal, offline, active-localization algorithms using the Fisher Information Matrix dates to Hammel et al., [14], and has seen more recent results by Logothetis et al. [15], Bishop et al. [16], [17] and Martinez et al. [18]. Of these, only [14] considered time-constrained trajectories. However, the results were for a single robot with a continuous sensor and so are not directly applicable to the setup considered in the present work.

We study the problem of including communication constraints. What separates this paper from other research is we consider if, and when, it is optimal to communicate. The problem of estimating the target state despite loss of connectivity has recently gained attention. Hollinger et al. [19] considered the problem of re-establishing an estimation task after losing connectivity. In the same vein, Makarenko et al. [20] studied estimation when connectivity was either enforced, or intermittent. This was similar to Leung et al. [21], who showed how to maintain a consistent estimate of a multi-robot system while relying on future reconnection. Spletzer et al. [22] studied the problem of assigning robots to targets while also enforcing network connectivity. In these works the optimality of maintaining connectivity was assumed, but we instead address establishing it directly.

IV. The Optimal Off-line Algorithm

We will first consider the *offline* case. Let the true target be $x^\star$. In the offline problem, $x^\star$ is known, and fixed. The goal is to design a minimum-cost measurement strategy S which satisfies the information requirements in Eq. (4). We will first study the case with unbounded communication range, then show how the solution changes as the communication range $r_c \to 0$ in Section IV-A. We will show, through a progressive series of proofs, the optimal sequence has a simple structure: It consists of only one measurement *location* per robot. We will also show how to find the optimal number of measurements to take at each location. First, we define the specific coordinate frame and notation we will be using in our proofs.

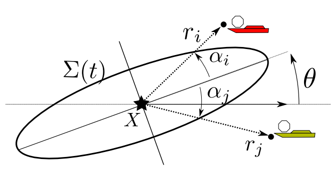


Fig. 2. The *target-local* coordinate frame. By expressing the measurement locations (black dots) with respect to the frame rotated by θ, we can operate on a diagonal matrix (the covariance ellipse's eigenvectors are aligned with the frame in which we express sensor locations).

Definition 1 (Target-Local (TL) frame): Define a coordinate frame, centered at $x^\star$. The rotation of the frame is specified by the orientation of the covariance of the target estimate at the current time step, $\mathbb{F}^{-1}(S)$. We align the x axis with the eigenvector corresponding to the maximum eigenvalue of $\mathbb{F}^{-1}(S)$. Additionally, all sensor locations are specified in polar coordinates, as $s_u^i = (\alpha_i, r_i)$, where α is the angle formed from the x axis.

We illustrate the TL coordinate frame in Figure 2. In practice, we can obtain a TL coordinate frame by applying a de-correlating transform, given by either the Singular Value Decomposition or the Eigendecomposition of $\mathbb{F}$ [23]. It is known to have the form [24]:

$$\mathbb{F}_{\mathrm{TL}}(S) = R(\theta)\begin{bmatrix} \sum_{i=1}^{N} \frac{\sin^2(\alpha_i)}{r_i^2\sigma_s^2} & 0 \\ 0 & \sum_{i=1}^{N} \frac{\cos^2(\alpha_i)}{r_i^2\sigma_s^2} \end{bmatrix} R(\theta)^T \quad (6)$$

where $R(\theta)$ can be interpreted as the rotation matrix to the TL frame from the world frame. The matrix $\mathbb{F}(S)$ has properties which are easier to illustrate in the TL frame as follows.

Lemma 1 (Properties of $\mathbb{F}_{TL}$): Let S be a measurement sequence with respect to $x^\star$. Let each element of S be expressed in polar coordinates, (α_i, r_i) w.r.t., $x^\star$. The Fisher Information Matrix of the sequence, $\mathbb{F}(S)$, when viewed in the Target Local frame, satisfies all of the following,

1) The smallest eigenvalue is $\lambda_{\min} = \sum \frac{\sin^2(\alpha_i)}{r_i^2\sigma_s^2}$
2) The largest eigenvalue is $\lambda_{\max} = \sum \frac{\cos^2(\alpha_i)}{r_i^2\sigma_s^2}$
3) Let $S_u \subset S$ be the measurement locations with $\sin(2\alpha) > 0$, and $S_v \subset S$ be the measurement locations with $\sin(2\alpha) < 0$. Then

$$\sum_{i=1}^{|S_u|} -\frac{\sin(2\alpha_i)}{r_i^2} = \sum_{i=1}^{|S_v|} \frac{\sin(2\alpha_i)}{r_i^2} \quad (7)$$

Proof: To prove 1 and 2, note the eigenvalues of the diagonal matrix in Eq. (6) are simply the diagonal elements.

The third constraint enforces that the measurement locations are in separate quadrants. To see the third property. Note the off-diagonal elements of the FIM(S) are given by $\sum_{i=1}^{|S|} -\frac{\sin(2\alpha_i)}{r_i^2\sigma_s^2}$. When the FIM is diagonalized this sum must equal 0. Thus either all $\sin(2\alpha_i) = 0$, or some α_i are greater than zero, and some are less. By separating the summation into two sums as described, we have the constraint as required. ∎

Having defined the coordinate frame we will be using, we can proceed to characterize the optimal measurement sequence. Recall from Section II we are trying to minimize the maximum time taken by either of the two robots, subject to $\lambda_d \geq \lambda_{\min}$. Consider the *optimal* two-robot measurement sequence, which we call $S^\star$. Let the eigenvalues of $\mathbb{F}(S^\star)$ be $\lambda^\star_{\min}$ and $\lambda^\star_{\max}$. Since $S^\star$ is a valid solution we know both are no less than λ_d, the desired information. We also know that $S^\star$ minimizes the maximum time spent by the two robots. It is also clear both robots spend the same amount of time on their sequences, or the maximum cost could be reduced by re-distributing the work between the robots.

What is not clear is the placement and assignment of measurement locations. Let S_u and S_v be the optimal robot assignments such that $S^\star = \{S_u + S_v\}$. Let the optimal number of measurements be $|S_u| = N_u$ and $|S_v| = N_v$. To illustrate the problem more clearly, we will choose to place the measurements at only two locations. We will shortly see why this is a good choice. By Lemma 1, we have three constraints on the sensor locations. By specifying only two measurement locations, we reduce all of the constraints as follows,

$$\lambda^\star_{\min} = N_u^\star \frac{\sin^2(\alpha_u)}{r_u^2\sigma_s^2} + N_v^\star \frac{\sin^2(\alpha_v)}{r_v^2\sigma_s^2} \quad (8)$$

$$\lambda^\star_{\max} = N_u^\star \frac{\cos^2(\alpha_u)}{r_u^2\sigma_s^2} + N_v^\star \frac{\cos^2(\alpha_v)}{r_v^2\sigma_s^2} \quad (9)$$

$$N_u^\star \frac{\sin(2\alpha_u)}{r_u^2} = -N_v^\star \frac{\sin(2\alpha_v)}{r_v^2} \quad (10)$$

We can show useful properties of these constraints. First, by solving for r_u (respectively r_v) in Eq. 8, we have an equation of the form $r = \pm C \cdot \sin(\alpha)$, which simply defines

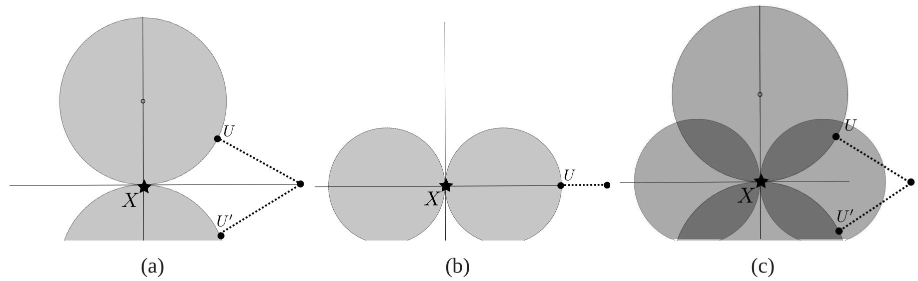


Fig. 3. An illustration of the constraints on the measurement sequences. (a) by Eq. 8, (b) by Eq. 9, and (c), the intersection. Note the radius in (b) is never greater than the radius in (a). Assuming the other measurements are placed, the last measurement U, must fall in the regions specified, while traveling the least. The dotted line adjoining U and the starting location illustrates the shortest path.

a pair of circles along the y-axis as shown in Figure 3(a). Similarly, Eq. 9 defines a pair of circles along the x-axis (See Figure 3(b)). To satisfy Eq (10), we specify oppositely signed α for the two measurement locations (i.e., one is above the x-axis, and one is below). What this means is the following: For any measurement sequence S_u and S_v for the first and second robot respectively, assuming S_u and S_v take all measurements from one location, the locations must lie in the intersection of the circles specified by Eqs. (8) and (9). We illustrate this in Figure 3(c).

Moreover, notice the convexity of the constraint induced by Eq. (8) (fig. 3(a)) implies $|\alpha_u| \leq 45°$. Otherwise, the cost to move to the measurement location is strictly increased. This implies satisfying the first constraint optimally will satisfy the second for free since $\cos(\alpha) > |\sin(\alpha)|$ for both sensor locations. Now, we will show the optimal algorithm also assigns only two measurement locations, possibly with more than one measurement taken at each location.

Lemma 2: Let $S^\star$ be the optimal measurement sequence. Let S_u be the measurement sequence for the first robot, and S_v be the second. Then both S_u and S_v contain only one measurement location.

Proof: Assume, to the contrary, that one or both of S_u or S_v contain more than one measurement location. Assume without loss of generality that we are dealing with S_u which takes all measurements with $\alpha_u > 0$. The same arguments can be applied to S_v and $\alpha_v < 0$. We will show this leads to a contradiction of the optimality of $S^\star$.

Refer again to the constraints imposed by $\mathbb{F}_{\mathrm{TL}}$, particularly Eq. (8). We have argued the optimal algorithm will not assign any $\alpha > 45°$. Optimizing the travel time for a single measurement location resulted in traveling directly to the closest point on the perimeter of the circle shown.

Now consider the problem of moving any measurement location away from the point S_u. Placing the measurement *inside* the circle will have a ratio $\frac{\sin(\alpha)}{r}$ *greater* than any point on the perimeter or outside. Since Eq. (8) is the scaled sum of these ratios, this implies an increase in one ratio produces an increase in $\lambda^\star_{\min}$. The converse is also true, reducing one ratio (moving a measurement outside the circle), lowers $\lambda^\star_{\min}$.

Therein lies the contradiction: by moving the location to the interior of the circle we cause Eq 8 to increase while traveling more. Thus, we are doing more work than necessary by traveling too much. Similarly, by moving the location along the perimeter, we add work unnecessarily. Finally, if we place any measurement outside the circle, we do not satisfy the constraints. ■

The previous result shows we can always construct a strategy with only two measurement *locations*: one per robot. From these locations, however, the robots may take as many measurements as is required. Next we will show we can always choose two measurement locations which are *symmetric* about the x-axis. For example, in Figure 3(a), U and U' are considered symmetric measurement locations.

Theorem 1 (One Step): There exists an optimal measurement sequence $S^\star$ which achieves $\lambda_d \leq \lambda^\star_{\min} \leq \lambda^\star_{\max}$ and consists of two measurement locations S_u and S_v, one for each robot. S_u and S_v are taken from the same range to the target, and have $\alpha_u = -\alpha_v$. At S_u and S_v, the same number of measurements are taken by each robot.

Proof: We will prove by construction. Let S' be any other optimal measurement sequence. We know by Lemma 2 S' has only two measurement locations, S_u and S_v, with N_u and N_v measurements taken respectively at these locations. We know the cost to travel to S_u and take N_u measurements is the same as the cost to travel to S_v and take N_v measurements. Finally, we know $|\alpha_u| < 45°$ and $|\alpha_v| < 45°$.

Consider Eq. (8) in the context of these two locations, S_u and S_v. Select the greater of the two terms on the right hand side; let this be the term corresponding to S_u, for example. Then form the sequence S'_v by setting α_v to $-\alpha_u$ and r_v to r_u. We have *increased* the information about $\lambda^\star_{\min}$. We have not lost much from $\lambda_{\max}$, since $\alpha < 45°$ implies $\cos(\alpha) \geq \sin(\alpha)$. Finally, we still satisfy Eq. (10) since $\alpha_u = -\alpha_v$. Moreover, we have not increased the total time cost since both tours took the same time to execute. Thus we have constructed an optimal symmetric sequence from an arbitrary optimal sequence and shown their equivalence. ■

We show an example of the optimal sequence in Figure 4. Theorem 1 shows a symmetric strategy can solve the original problem optimally, but we have not shown how to construct the symmetric strategy directly. To compute the optimal strategy, we will optimize the cost of a sequence as a function of the N measurements taken. Since we know Eq. (8) holds with equality and $N_u = N_v$, we know the optimal algorithm must travel *exactly* to the circles given by radius, as a function of N,

$$r_\Lambda(N) = \frac{1}{2}\sqrt{\frac{N}{\lambda_d \sigma_s^2}} \tag{11}$$

given $N \geq 2$ is $N_u + N_v$. Note each robot only takes $1/2$ the measurements, but is responsible for $1/2$ the summation in Eq. 8. Thus, the factor two cancels on the right-hand side. Let the robots start at a distance, $d_\star$ from $x^\star$, the cost of both of the sequence $S = \{S_u, S_v\}$ is then (see Figure 4).

$$C(S) = \sqrt{d_\star^2 + r_\Lambda^2(N)} - r_\Lambda(N) + \frac{N}{2}t_m \tag{12}$$

We can now define a simple measurement strategy, which we refer to as the `ONE-STEP` algorithm. First, we minimize equation (12) with respect to N to find the optimal number of measurements as a function of the sensor noise, measurement cost, and desired uncertainty. We then substitute this value into Eq. 11 to find the optimal circle. Sending one robot to the closest point on the circle *above* the x-axis, and one to the circle below, then taking $\frac{N}{2}$ measurements each will solve the objective in minimal time. The algorithm for computing the offline-optimal sequence, which we call `ONE-STEP`, is presented in Algorithm 1.

Algorithm 1 $\{S_u, S_v\} \leftarrow$`ONE-STEP`$(s_0, x^\star, \lambda_d, t_m)$

$N_\star \leftarrow$ argmin$_N$ Eq. (12)
$r_\Lambda^\star \leftarrow$ Evaluate Eq (11) with $(N_\star)$
$\alpha_\star, r_\star \leftarrow$ closest point on the circle defined by $r_\Lambda^\star$
$S_u \leftarrow \frac{N}{2}$ measurements at $(\alpha_\star, r_\star)$
$S_v \leftarrow \frac{N}{2}$ measurements at $(-\alpha_\star, r_\star)$

A. With Communication Constraints

We have not yet considered the effect of communication on the optimal strategy. We can introduce communication constraints as follows. One possible strategy is simply to execute Algorithm 1, then move toward the centroid of the robot's positions until within communication range. We illustrate this strategy in Figure 4, as the black line. However, it may be more time-efficient to simply move to s_u', which places the robots in communication range during measurements (as illustrated by the dotted line). We easily express the cost as a function of the measurement location, then optimize as before. Expand Eq. (12) to include a rendezvous cost. As a function of the N measurements needed, the cost becomes,

$$\sqrt{d_\star^2 + r_u^2 - 2d_\star r_u \cos(\alpha_u)} + \frac{N}{2}t_m + \min(r_u \sin(\alpha_u) - \frac{1}{2}r_c, 0) \tag{13}$$

The cost clearly has a single minimum for any given N since $\alpha \in [0, \frac{\pi}{4}]$. The minimum corresponds to the optimal measurement location when constrained by communications. We formalize these arguments with the following

Algorithm 2 $\{S_u, S_v\} \leftarrow$`TWO-STEP`$(s_0, x^\star, r_c, \lambda_d, t_m)$

$N_\star, r_\star, \alpha_\star \leftarrow$ argmin$_{N,\alpha,r}$ Eq. 13.
$S_u \leftarrow \frac{N}{2}$ measurements at $(\alpha_\star, r_\star)$
$S_u \leftarrow \{S_u \bigcup (r_\star \cos\alpha_\star, \frac{r_c}{2})\}$
$S_v \leftarrow \frac{N}{2}$ measurements at $(-\alpha_\star, r_\star)$
$S_v \leftarrow \{S_v \bigcup (r_\star \cos\alpha_\star, -\frac{r_c}{2})\}$

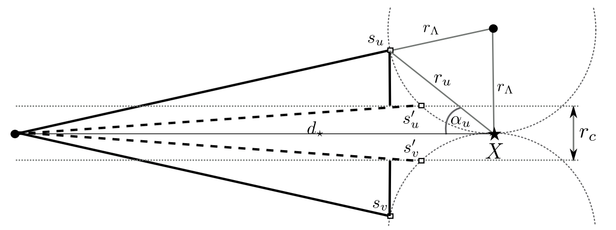


Fig. 4. An illustration of two different choices for communication-constrained measurement locations. s_u and s_v represent the output of `ONE-STEP`. From these locations, the robots can move directly toward each other to communicate. Alternatively, they can move to s_u' and s_v', and remain in communication during measurement. The `TWO-STEP` algorithm finds the optimal placement of measurements to minimize the cost while including the rendezvous cost.

algorithm. The algorithm is simply a generalization of the `ONE-STEP` algorithm: as $r_c \to \infty$ the output is the same.

Theorem 2: The algorithm `TWO-STEP` is the optimal symmetric strategy for the offline problem with limited communication range.

Proof: The proof proceeds similarly to the proof of Theorem 1, but with the cost of the sequence changed. ■

These strategies (perhaps counter-intuitively) take all of their measurements from a single location (per robot). What we have shown is the following. First, there is an optimal measurement location for a given target location. Second, if the target location is not expected to change, then the robots need simply move to the optimal location and take measurements until the information objective is satisfied. We now move on to the *online* case, when each measurement is likely to change the estimate of the target location, and so the robots must be careful not to spend too much time moving towards a distant target estimate if it is highly likely to change given a few measurements.

V. Online Algorithm

In this section we present an *online* variant of the previous strategies. In the online setting, which much more closely models real-world scenarios, we do not have access to the true target location $x^\star$. Instead, we are given an estimate in the form of a prior probability density function (PDF).

One possible extension to the offline algorithm is to choose the most likely point in the PDF to be $x^\star$, then execute the `TWO-STEP` algorithm with respect to this point. Since we deal with Gaussian target estimates, the most likely point is the mean, $\hat{x}$. However, the true target location may be close to the robots' initial location, while $\hat{x}$ may be relatively far away. Then `TWO-STEP`$(\hat{x})$ will do significantly more work than `TWO-STEP`$(x^\star)$. Given the symmetry of a Gaussian PDF, approximately half of the time $x^\star$ is closer than $\hat{x}$.

Instead, we will choose the *closest* point of the current PDF with "high enough" probability. For example, if we wanted to ensure we accounted for 99 percent of all the probability mass, we could find the contour of the $3-$sigma bounds of the two-dimensional Gaussian estimate at each time-step, and use the closest point on this contour.

We will proceed as follows. At each step, form a circle which completely contains the $3-$sigma ellipse. Then, we find the closest point on the circle, which we label x_a. We can execute the TWO-STEP algorithm on x_a, paying cost $C(x_a)$. Then, when we have gathered measurements from near x_a, we update the hypothesis and select a new x_a from the posterior PDF. We can now present a formalized algorithm built on this intuition, shown in Algorithm 3.

To bound the cost of Algorithm 3, we will argue two things. First, we will show the time required by each execution of the TWO-STEP subroutine is no greater than the time required by the optimal algorithm to localize the target. Then, we will show that a limited number of calls to TWO-STEP are required to localize the target. This will produce the bound we present in Theorem 3.

Lemma 3: Let $\mathcal{C}_i$ be a circle around $\hat{x}$ at time i, such that $P(x^\star \in \mathcal{C}_i) \to 1$. Let x_a be the closest point on the current circle to the robots' starting location. Let C_i be the cost of the i^{th} call to TWO-STEP(x_a) and the optimal cost $C^\star$ be $C(\texttt{TWO-STEP}(x^\star))$. Then all costs, C_i are less than $C^\star$ with probability close to 1.

Proof: The difference in costs TWO-STEP(x_a) and TWO-STEP$(x^\star)$ is determined by the distance between the starting location and the target point (x_a or $x^\star$).

Let s_0 be the starting location of the robots. Let s_i be the starting location for the i^{th} call to TWO-STEP(x_a). We will show $||s_i - x^\star|| \le ||s_0 - x^\star||$ for all i. The proof is illustrated in Figure 5.

At the first step, we travel distance $||s_0 - x_a||$. Since x_a is on the first circle it is closer to the starting location than $x^\star$ by definition. Now consider any other placement of the next circle. By assumption, their intersection must contain $x^\star$. Since their intersection is convex, and the circles themselves are convex, each time we move to the boundary of a circle, we decrease the distance to their mutual intersection. Thus, at each step $||s_i - x^\star|| \le ||s_0 - x^\star||$. Since the distance to the target point is less than the starting distance, and the radius of the circles decrease at each step, the statement of the lemma follows. ∎

We have shown we do not pay more than the optimal algorithm at any step. However, it may take many calls to TWO-STEP to reduce the uncertainty adequately, producing an arbitrarily high cost compared to the optimal algorithm. However, we can show this is not the case as follows.

Lemma 4: The MULTI-STEP algorithm requires $\mathcal{O}(\log \lambda_d - \log \lambda_{\min}(\mathbb{F}(0)))$ calls to TWO-STEP.

Proof: We will show a lower bound on the amount of information gathered at each step of the MULTI-STEP algorithm. First note by the properties of FIM we have $\mathbb{F}(i+1) = \mathbb{F}(i) + \mathbb{F}(s_u, s_v)$ for measurements taken at the sensor locations s_u, s_v.

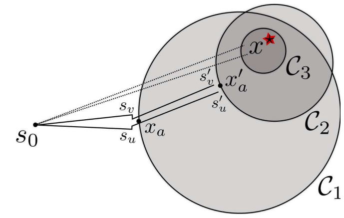


Fig. 5. An illustration of the two-robot MULTI-STEP algorithm (solid paths) and the optimal TWO-STEP algorithm (dashed paths). The robots begin at location s_0. The optimal choice is to move directly to $x^\star$, but we are unsure of the location of $x^\star$. Thus, we form a circle $\mathcal{C}$ such that the probability $x^\star$ is in $\mathcal{C}$ is high. At each step of the MULTI-STEP algorithm, we do less work than the optimal TWO-STEP by Lemma 3. Since the number of TWO-STEP executions is bounded by Lemma 4, we do not do significantly more work than optimal by Theorem 3

Algorithm 3 MULTI-STEP$(s_0, \hat{x}(0), \Sigma(0), r_c, \lambda_d, t_m)$

```
Σ(i) ← Σ(0)
x̂(i) ← x̂(0)
while λ_min(Σ(i)^{-1}) ≤ λ_d do
    C_i ← circle of radius 3 · sqrt(λ_max(Σ(i))) at point x̂(i)
    x_d ← closest point on C_i
    s_i ← centroid of robot locations
    TWO-STEP(s_i, x_d, r_c, λ_d, t_m)
    Z ← Collect measurements
    x̂(i), Σ(i) ← Update target estimate using Z
end while
```

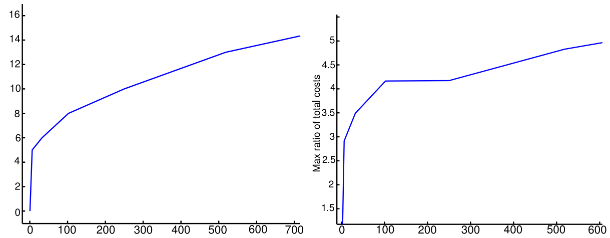
Recall Eq. 8. Let β be the angle $s_u - x^\star - s_v$, and $r_\star$ be the distance to the true target. $\mathcal{C}_i$ is the circle of diameter $|\mathcal{C}_i|$ which encompasses the desired probability mass of the PDF of the target estimate. Then $|\mathcal{C}_i| \propto \sqrt{\lambda_{\max}(\mathbb{F}(i)^{-1})} = \sqrt{\frac{1}{\lambda_{\min}(\mathbb{F}(i)}}$. Note the maximum range to the true target is at most the diameter of the circle, and there is no point in the circle which is parallel with the two robots during their measurements. Thus, $\sin\beta > 0$ and $r \le |\mathcal{C}_i|$. By substituting these values into Eq. (8), we have

$$\begin{aligned}\lambda_{\min}(\mathbb{F}(t+1)) &\ge \lambda_{\min}(\mathbb{F}(t)) + \frac{\sin^2\beta}{\sigma_s^2}\lambda_{\min}(\mathbb{F}(t)) \\ &\ge \lambda_{\min}(\mathbb{F}(t))\left(1 + \frac{\sin^2\beta}{\sigma_s^2}\right) \qquad (14)\end{aligned}$$

which implies at least a constant factor increase in information (and corresponding decrease in uncertainty) at every time step. If MULTI-STEP makes $N \ge 1$ calls to TWO-STEP, then $\lambda_{\min}\mathbb{F}(N) = \lambda_{\min}(\mathbb{F}(0))\left(1 + \frac{\sin^2\beta}{\sigma_s^2}\right)^N$. Solving implies $N \le \log_b(\lambda_{\min}(\mathbb{F}(0)) - \log_b(\lambda_{\min}(\mathbb{F}(N)))$ calls to the TWO-STEP subroutine, with $b > 1 \ge \left(1 + \frac{\sin^2\beta}{\sigma_s^2}\right)$. ∎

Our final result follows: An upper bound on the worst-case ratio of work done by our MULTI-STEP algorithm versus the optimal algorithm.

Theorem 3 (Cost Bounds): The ratio of the cost of the MULTI-STEP algorithm to the optimal offline algorithm satisfies $\frac{\texttt{MULTI-STEP}(\hat{x}(0))}{\texttt{TWO-STEP}(x^\star)} = \mathcal{O}(\log \lambda_d - \log \lambda_{\min}(\mathbb{F}(0)))$



(a) Number of calls to `TWO-STEP` (b) Ratio of costs

Fig. 6. The aggregate results of numerical studies. Left: the number of calls to `TWO-STEP` as a function of $\frac{\lambda_d}{\lambda_{\min}(\mathbb{F}(0))}$. Right: the ratio of costs of online algorithm to optimal offline algorithm, as a function of $\frac{\lambda_d}{\lambda_{\min}(\mathbb{F}(0))}$. Shown is the maximum value encountered during simulations.

Proof: By Lemma 4 we make $\mathcal{O}(\log \lambda_d - \log \lambda_{\min}(\mathbb{F}(0)))$ calls to `TWO-STEP`, and by Lemma 3 we know each of these costs is less than the optimal cost, the result follows. ■

In this section we have shown an adaptive, online algorithm to localize a target given only a prior estimate of its location. Theorem 3 shows the `MULTI-STEP` algorithm requires a logarithmic factor more calls to `TWO-STEP` than the optimal algorithm. In the next section we explore the results of Theorem 3 in simulations.

VI. Implementations and Experiments

Here we explore the results of Theorem 3 through simulations and experiments. Our goal is to verify the logarithmic behavior of the bound presented in the previous section.

A. Simulations

We explore the effects of the ratio of prior information to desired precision through simulations. The results are presented in Figure 6. The x-axis of the figures shows the ratio $\frac{\lambda_d}{\lambda_{\min}(\mathbb{F}(0))}$ (desired gain in information). The ratio of prior uncertainty to final uncertainty is the inverse of this.

We fix a prior target estimate and starting robot location along the edge of the estimate. The desired precision, λ_d was held fixed. The starting precision, $\lambda_{\min}(\mathbb{F}(0))$, ranged from λ_d to $.0015 \cdot \lambda_d$. Then, for each value of $\lambda_{\min}(\mathbb{F}(0))$, we repeatedly test the performance ratio by sampling a true target location from the prior PDF and executing `TWO-STEP` using the true target location, and `MULTI-STEP` on the hypothesis. To give a real-world sense of scale to the simulations, note our choice represents a starting hypothesis which grows to encompass a 254 square kilometer area, while requiring a final estimate which is as accurate as a commercial GPS fix (i.e., a few meter uncertainty).

First, we present the actual number of calls the `MULTI-STEP` algorithm makes to the `TWO-STEP` subroutine. We notice a logarithmic trend to the ratios, as expected. These results are shown in Figure 6(a). The number of calls is not necessarily reflective of the ratio of the costs between our online algorithm and the optimal offline algorithm, since the total distance traveled for each call will decrease.

To explore this, we present the ratio of the *actual* cost in Figure 6(b). The actual cost is given by the maximum distance travel-led plus the maximum time spent measuring. Because we expect the cost ratio to significantly change, depending on the relative positions of the true target location, hypothesis location and uncertainty, and the starting robot positions, in Figure 6(b) we present the worst-case ratio of costs, for each prior hypothesis. Interestingly, the worst-case ratio of costs was less than 7 in these trials, much less than the number of calls to the `TWO-STEP` subroutine. This suggests in practice the cost of our online algorithm is closer to the optimal offline algorithm than is suggested by the theoretical results.

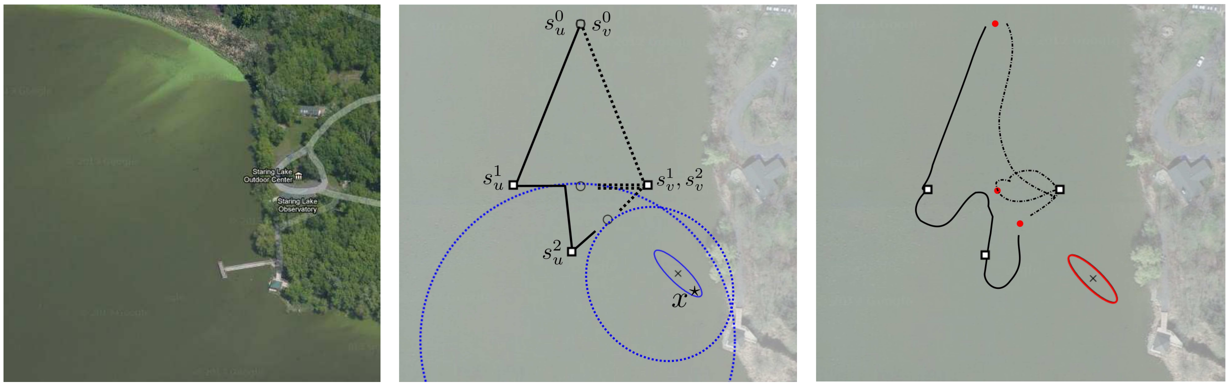
B. Field Experiments

As mentioned, we are building a working implementation to search for invasive fish. To test the suitability of the algorithm to real-world conditions, we have implemented the algorithm for field trials on lakes in Minnesota, USA. We report a field experiment which confirms the feasibility of the algorithm in practice. The robots used were OceanScience QBoats, pictured in Figure 1. The boats are designed for autonomous tracking of radio-tagged fish. They are 2 meters in length, have an average speed of 1 meter per second, and take approximately 1 minute to measure a bearing to a transmitting tag.

The experiment was run in Lake Staring, MN, USA (shown in Figure 7). A transmitting tag was deployed at a known location in the environment, and the robots executed the `MULTI-STEP` algorithm. The boats began 140 meters from the target, and executed the algorithm given in Section V. To exchange measurements, the robots met and transmitted over an ad-hoc wireless network. The final result is shown in Figure 7(c), as a blue square. We show the algorithm steps in Figure 7(b), and the actual robot paths in Figure 7(c). After each measurement, the boats transmitted measurement values over the wireless network. We used a low power network which could not communicate more than 10-20 meters reliably. Over the experiment shown, the robots traveled a combined distance of one kilometer and localized the target to within 10 meters of its reported location. Note the position of the tag's location was accurate to within 3 meters due to GPS error. The localization took only 2 steps to locate the target. We provide a video of the localization process at [25], and also accompanying this submission.

VII. Conclusions

We have presented three main results. First, we showed the first derivation of the optimal deployment of two mobile bearing sensors with respect to a known target location. The result was extended to provide the optimal symmetric deployment when subjected to limited communications. The insights provided by these results were used to develop an online strategy suitable for field deployments. We showed a theoretical upper bound of a logarithmic approximation of the optimal strategy. To further reinforce the results, we provided simulations studies verifying the bounds presented.



(a) The experiment area. (b) The execution steps of the algorithm. (c) The robot paths overlaid.

Fig. 7. Experiment results from Lake Staring, MN, USA. **(a)**: The experiment area. The true target (and camera in [25]) were placed on the docks near the bottom right corner. The robots began near the top-middle. **(b)**: The two calls to `TWO-STEP` produced the dark paths shown, and reduced the uncertainty (the blue circles). The final actual uncertainty was the solid ellipse. **(c)**: More execution details. The solid red circles are the points where the robots exchanged information.

Finally, we showed a working field deployment over a large operating environment. In this example, two robots were able to locate a radio transmitter within 10m.

Our future work will focus on tightening the logarithmic approximation of our proposed online algorithm. We expect that a constant factor approximation is possible. Another avenue for future work is reducing the number of communication steps. Design and analysis of active localization algorithms for multiple robots in also an important avenue for future work.

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REFERENCES

[1] P. Tokekar, D. Bhadauria, A. Studenski, and V. Isler, “A Robotic System for Monitoring Carp in Minnesota Lakes,” *Journal of Field Robotics*, vol. 27, no. 6, pp. 779–789, 2010.

[2] Y. Bar-Shalom, X.-R. Li, and T. Kirubarajan, *Estimation with Applications to Tracking and Navigation*. New York, USA: John Wiley & Sons, Inc., 2001.

[3] H. L. Van Trees, *Detection, Estimation and Modulation Theory*. New York, NY: John Wiley & Sons, Inc, 2001.

[4] B. Grocholsky, “Information-theoretic control of multiple sensor platforms,” Ph.D. dissertation, University of Sydney. School of Aerospace, Mechanical and Mechatronic Engineering, 2006.

[5] B. Grocholsky, A. Makarenko, and H. Durrant-Whyte, “Information-theoretic coordinated control of multiple sensor platforms,” in *Robotics and Automation, 2003. Proceedings. ICRA’03. IEEE International Conference on*, vol. 1. IEEE, 2003, pp. 1521–1526.

[6] E. Frew and S. Rock, “Exploratory motion generation for monocular vision-based target localization,” vol. 7, 2002, pp. 3633–3643.

[7] E. W. Frew, “Observer trajectory generation for target-motion estimation using monocular vision,” Ph.D. dissertation, Stanford University, 2003.

[8] K. Zhou and S. Roumeliotis, “Multirobot Active Target Tracking With Combinations of RelativeObservations,” *Robotics, IEEE Transactions on*, vol. 27, no. 4, p. 678, 2011.

[9] P. Tokekar, J. Vander Hook, and V. Isler, “Active target localization for bearing based robotic telemetry,” in *Intelligent Robots and Systems (IROS), 2011 IEEE/RSJ International Conference on*. IEEE, 2011, pp. 488–493.

[10] P. Tokekar, E. Branson, J. Vander Hook, and V. Isler, “Coverage and Active Localization for Monitoring Invasive Fish with an Autonomous Boat,” *IEEE Robotics and Automation Magazine*, 2013, in press.

[11] J. Vander Hook, P. Tokekar, E. Branson, P. Bajer, P. Sorensen, and V. Isler, “Local-search strategy for active localization of multiple invasive fish,” in *International Symposium on Experimental Robotics*, 2012.

[12] J. Derenick, J. Fink, and V. Kumar, “Localization Using Ambiguous Bearings from Radio Signal Strength,” *2011 IEEE/RSJ International Conference on Intelligent Robots andSystems*, pp. 3248–3253, Sept. 2011.

[13] C. Forney, E. Manii, M. Farris, M. Moline, and C. Lowe, “Tracking of a Tagged Leopard Shark with an AUV: Sensor Calibrationand State Estimation,” in *IEEE International Conference on Robotics and Automation*, 2012, pp. 5315–5321.

[14] S. Hammel, P. Liu, E. Hilliard, and K. Gong, “Optimal observer motion for localization with bearing measurements,” *Computers & Mathematics with Applications*, vol. 18, no. 1, pp. 171–180, 1989.

[15] A. Logothetis, A. Isaksson, and R. J. Evans, “An information theoretic approach to observer path design for bearings-only tracking,” in *Decision and Control, 1997., Proceedings of the 36th IEEE Conference on*, vol. 4. IEEE, 1997, pp. 3132–3137.

[16] A. N. Bishop, B. Fidan, B. D. Anderson, K. Doançay, and P. N. Pathirana, “Optimality analysis of sensor-target localization geometries,” *Automatica*, vol. 46, no. 3, pp. 479–492, Mar. 2010.

[17] A. N. Bishop and P. N. Pathirana, “Optimal trajectories for homing navigation with bearing measurements,” in *Proceedings of the 2008 International Federation of Automatic Control Congress*, 2008.

[18] S. Martinez and F. Bullo, “Optimal sensor placement and motion coordination for target tracking,” *Automatica*, vol. 42, no. 4, pp. 661–668, Apr. 2006.

[19] G. Hollinger and S. Singh, “Multi-robot coordination with periodic connectivity,” in *Robotics and Automation (ICRA), 2010 IEEE International Conference on*. IEEE, 2010, pp. 4457–4462.

[20] A. Makarenko and H. Durrant-Whyte, “Decentralized data fusion and control in active sensor networks,” in *Proceedings of the Seventh International Conference on Information Fusion*, 2004.

[21] K. Y. K. Leung, S. Member, T. D. Barfoot, and H. H. T. Liu, “Decentralized Localization of Sparsely-Communicating Robot Networks : A Centralized-Equivalent Approach,” vol. 26, no. 1, pp. 62–77, 2010.

[22] J. R. Spletzer and C. J. Taylor, “Dynamic sensor planning and control for optimally tracking targets,” *The International Journal of Robotics Research*, vol. 22, no. 1, pp. 7–20, 2003.

[23] G. H. Golub and C. F. V. Loan, *Matrix computations (3rd ed.)*. Johns Hopkins University Press, 1996.

[24] J. Vander Hook, P. Tokekar, and V. Isler, “Cautious Greedy Strategy for Bearing-Only Active Localization: Analysis and Field Experiments,” *Journal of Field Robotics*, 2013, under review.

[25] “Two-robot field experiment,” 2013. [Online]. Available: https://www.youtube.com/watch?v=jZWTUxqTskY