Coverage path planning under the energy constraint

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Revised
**Coverage Path Planning Under the Energy Constraint**

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*Abstract*—In coverage applications, a common assumption is that the robot can fully cover the environment without recharging. However, in reality most mobile robot systems operate under battery limitations. To incorporate this constraint, we consider the problem when the working environment is large and the robot needs to recharge multiple times to fully cover the environment.

We focus on a geometric version where the environment is represented as a polygonal grid with a single charging station. Energy consumption throughout the environment is assumed to be uniform and proportional to the distance traveled. We first present a constant-factor approximation algorithm for contour-connected environments. We then extend the algorithm for general environments. We also validate the results in experiments performed with an aerial robot.

**I. INTRODUCTION**

Coverage is a fundamental robotics problem. In many practical coverage applications, energy limitations would prevent a robot from covering the entire environment in one iteration. Before the battery runs out, the robot has to visit a charging station to get fully recharged to continue its work. As a result, instead of a single path, we may need to plan multiple paths to cover the environment. Each path should start and end at a charging station. The robot’s energy budget should be enough to follow each path without recharging. In this paper, we study the coverage path planning problem with this energy constraint.

The literature on coverage problem can be divided into two groups based on the environment representation. (i) The environment is represented as a graph. Without the energy constraint, the coverage problem becomes the well-known Traveling Salesperson Problem (TSP). The Vehicle Routing Problem (VRP) considers the energy and capacity constraints when visiting vertices. One version is called Distance Vehicle Routing Problem (DVRP), which models the energy consumption proportional to the distance traveled. The VRP is mainly solved by integer programming and heuristic methods [9]. Nagarajan considered DVRP on tree metrics and proposed a 2-approximation algorithm [12]; (ii) The second common representation of the environment is a polygon. The robot is represented as a unit disk or square. The goal is to move the robot in the polygon so that every point in the polygon is covered. The choice of the representation depends on the practical application.

Coverage path planning problem under energy constraints has been studied extensively in the literature [7]. Well-known coverage strategies include the boustrophedon decomposition coverage with the back-and-forth motion [4], spiral path coverage [8], and spanning-tree based coverage [6]. The boustrophedon coverage and spiral path coverage can be adapted to perform in an online fashion by tracing the uncovered area [14] [3]. The objective of these problems is to completely cover the environment. Yoav and Elon restrict the robot to move rectilinearly, the algorithm is optimal when there is no zero-thickness in the spanning tree [6].

Coverage with energy constraints is a relatively new topic. Shnaps and Rimon [13] studied this problem and modeled the energy cost by the path length. They provided an approximation algorithm with a factor of \( \frac{1}{1-\rho} \), where \( \rho \) is the ratio between the distance of the furthest cell from recharging station and half of the energy budget. This factor can be arbitrarily large when \( \rho \) approaches 1.

Our contributions: We revisit the setting in Shnaps and Rimon [13] and present a constant-factor approximation algorithm for the energy-constrained coverage problem for equi-distance-contour-connected environment (contour-connected for short) when the robot is restricted to axis-parallel motion. We then generalize this algorithm to arbitrary polygons by partitioning a given polygon into contour-connected pieces. Compared with prior work, the deviation of the cost of our algorithm from the optimal cost remains bounded. We also present the results from a field experiment performed with an autonomous aerial vehicle.

**II. PROBLEM FORMULATION**

We are given an environment represented as a unit-grid laid out on a polygon \( P \). \( P \) has a single charging station \( S \) in it. The robot is represented as a unit square which can only move rectilinearly in \( P \). The robot’s energy consumption is assumed to be proportional to the distance traveled. We use 'path length' and 'energy cost' interchangeably. Due to the energy constraint the robot can only move at most \( B \) units after a full charge. Then it has to get recharged at \( S \). The goal is to find a set of paths, \( \Pi = \{ \pi_1, \pi_2, \ldots, \pi_n \} \), for the robot such that \( \bigcup_{i=1}^{n} \pi_i = P \) and the number of paths in \( \Pi \) is minimized. In addition, each path should start and end at \( S \) and \( |\pi_i| \leq B \) for \( i = 1, 2, \ldots, n \).

A related goal is to minimize the total length of the paths in \( \Pi \). In our analysis, we bound the number of paths first, then bound the total length.

**III. A GENERAL TSP BASED ALGORITHM**

Before we present our constant-factor approximation algorithm, we first show how TSP-partitioning techniques (see, e.g. [2], [5]) can be used to obtain a simple algorithm which matches the performance of [13].

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Let $r$ be the radius of the environment (the furthest distance from $S$). We must have $B = 2r + \alpha$ with $\alpha \geq 0$ because otherwise the robot does not have enough battery to reach the furthest cells. Let $T$ be the optimal TSP tour of the grid cells (or vertices), and OPT be the optimal number of battery-constrained paths to cover the environment. We have $OPT \geq T/B$.

We now partition $T$ into $k = T/(B - 2r)$ segments (when $\alpha > 0$) and form subtours by connecting the beginning and the end of each segment to $S$. Each subtour has length at most $B$. The ratio of the total number of subtours to the optimal tour is at most

$$\frac{T/(B - 2r)}{T/B} = 1 + \frac{2r}{\alpha}$$

A few remarks are in order: Since TSP is NP-Complete, we cannot compute the optimal tour $T$. However, in Euclidean settings, we can use a PTAS to obtain an arbitrarily close solution [1]. Second, this argument can apply to general environments, e.g. with non-uniform costs as long as the environment can be represented as a graph. Furthermore, if the robot’s energy budget is large enough, e.g. $B > 3r$, this yields a 3-approximation algorithm. However, in theory $\alpha$ can be arbitrarily small. Just like the algorithm in [13] [10], the theoretical performance of this technique can be arbitrarily bad.

In the rest of the paper, we deal with this issue and show how a depth-first-search-like coverage strategy yields a 4-approximation algorithm in certain environments.

IV. PRELIMINARIES

In this section, we present the definition of contour-connected environments and introduce several terms and notations which are necessary for our analysis.

An equi-distance contour (contour for short) is a polyline the cells on which have the same distance to the charging station. A cell is called a split cell if this cell is adjacent to the boundary and the equi-distance contour splits into two segments at this cell, as defined by Shnaps and Rimon [13]. The contour-connected environment refers to an area without split cells. Note that the cells on the outer most contours are not considered to be split cells though they are adjacent to the boundary. Any convex environment with a charging station at an acute vertex is contour-connected. Fig. 1 shows three examples, two of which are contour-connected and one is not. Second and third examples illustrate that the position of the charging station $S$ affects whether the environment is contour-connected or not.

When covering the environment, if any cell on a contour $C$ is visited multiple times, we call $C$ a saturated contour. $C$ denotes the area further than $C$ to $S$. For two cells $p$ and $q$, if any shortest path from $S$ to $q$ passes through $p$, we say $q$ is accessible from $p$. Let $C_{sub}$ be a subsegment of $C$. Then subarea $C_{sub}$ is the union of all the cells in $C$ which are accessible from the cells on $C_{sub}$, as shown in Fig. 2. We use $d(\cdot)$ to denote the distance from the contour or the cell to $S$, and use $|\ |$ to denote the number of the cells on the contour or in the area.

A contour-connected environment has the following properties. The proofs are presented in the appendix.

**Proposition 1:** If $C$ is an equi-distance contour in a contour-connected environment, then $C$ is a line segment.

**Proposition 2:** Let $C$ and $C'$ be two adjacent equi-distance contours. Then $||C| - |C'|| \leq 1$.

**Proposition 3:** Let $C$ and $C'$ be two contours in a contour-connected environment. Suppose $C'$ is further than $C$. $c_1, c_2, ..., c_p$ are the cells on $C$ from one side to the other. $c_1', c_2', ..., c_q'$ are the cells on $C'$ from the same side to the other. ($c_i$ is defined to be $c_p$ if $i > p$, and $c_i'$ is defined to be $c_q'$ if $i > q$). Then $c_i'$ is accessible from $c_i$ for all $i$.

V. COVERAGE STRATEGY FOR CONTOUR-CONNECTED ENVIRONMENTS

We now present our coverage algorithm for contour-connected environments and prove its performance. Let $r$ be the robot’s position. We define three types of motions for the robot. (i) Advance: For all the unvisited cells $c$ with $d(c) > d(r)$, the robot moves to the closest one along the shortest path. (ii) Follow: the robot follows the current contour first, then follow the closer uncovered contours to $S$. (iii) Retreat: The robot returns to $S$ along the shortest path. Note that the Advance motion may contain the Follow motion. Initially $r = S$. Re denotes the current remaining energy budget of the robot.

Algorithm 1 proceeds as follows. The robot first goes to the furthest uncovered cell in the environment using the Advance motion. Then it follows the contours by the Follow motion. When the remaining energy is just enough for going back to $S$, the robot retreats. For the Advance motion, there may be multiple candidate cells to move to. In
Algorithm 1 Coverage of contour-connected environments.

Output: $\Pi_{sol}$.

1: $\Pi_{sol} = \emptyset$; $i=1$;
2: Start recording path $\pi_i$;
3: while $P$ is not fully covered do
4:   Advance to the furthest uncovered cells.
5:   Follow until $Re = d(r)$.
6:   if $i$ is odd then
7:       record current position, $p=r$.
8:   end if
9:   Retreat. Recharge the robot; Add $\pi_i$ to $\Pi_{sol}$;
10: $i++$; Start recording path $\pi_i$;
11: if $i$ is even then
12:   Move to $p$ along shortest path;
13: end if
14: end while

this case, we choose the one adjacent to the boundary or the previous path to avoid splitting the environment. The strategy is presented in Algorithm 1, which essentially performs a depth-first-search-like coverage. An execution example is shown in Fig. 3.

To prove the performance of the algorithm, we need to compare $\Pi_{sol}$ to the optimal solution. We take an indirect approach. We first transform the optimal solution into a feasible solution in the following way. We split each path into the sub-paths to cover $\Pi_{sol}$ we show that if $\Pi$ is not fully covered

$\Pi$ that when the robot reaches any saturated contour $\Pi$ of the paths of the optimal solution. We compare $\Pi$ which fully covers the environment and has twice the number $\Pi$ of the paths in our solution.

Fig. 4 shows such an example. We get a path set, $\Pi_{sol}$ to visit at least $B/2$ cells. So we let an even-indexed path start from where the previous path retreats and set its coverage phase to start there (line 11 to 13). In our analysis, we pair them and have Proposition 4.

Proposition 4: Let $\pi_{2k+1}$ and $\pi_{2k+2}$ be a path pair from Algorithm 1. Their coverage phases together visit at least $B/2$ cells.

Proof: Let $\pi_{2k+1}$ retreat on the contour $C$. $\pi_{2k+2}$'s coverage phase visits $n_1 = d(C) + (B - d(C))/2$ cells. $\pi_{2k+2}$'s coverage phase visits $n_2 = (d(C') - d(C)) + (B - d(C'))/2$ cells. $n_1 + n_2 = B + (d(C') - d(C)) > B/2$, which means the coverage phases of $\pi_{2k+1}$ and $\pi_{2k+2}$ are able to visit at least $B/2$ cells. When $C'$ is closer than $C$, as shown in Fig. 5(b), we can get the same result.

For simplicity, we use $\Pi_{sol} = \{\pi_1, \pi_2, ..., \pi_m\}$ to denote the output of our algorithm, where each $\pi_i$ is a path pair from Algorithm 1. The coverage phases of these paths visit at least $B/2$ cells.

The $\Pi_{sol}$ is adjacent to the boundary. It enters the contours from one side. Without loss of generality, we call this side the left side. We have the following proposition.

Proposition 5: After every path in $\Pi_{sol}$, the covered cells
on any contour $C$ are all on the left side of $C$.

Proof: By Algorithm 1, the Advance motion of every path in $\Pi_{sol}$ enters any contour from the left side. There is one case when the covered cells on a contour are on the right side: A path starts to follow a contour from the right side, but it does not have enough energy to finish following this contour, as shown in Fig. 6. The robot uses the Follow motion to finish a contour $C'$ and enters the closer contour $C'$ from right side. Then the robot retreats from $C$ to $S$. When the robot follows the contour $C'$, its actual trajectory switches between $C$ and $C'$, as shown by the dashed line in Fig. 6. So if we count the intermediate cells when following $C'$ as covered, $C$ is fully covered. This is the only case when we count the intermediate cells as covered. Thus the covered cells are all on the left side of any contour.

Proposition 5 leads to the following proposition.

Proposition 6: If a path performs the Follow motion on a contour $C$, any contour $C'$ in $C$ (if exists) has uncovered cells no more than $C$ has.

Proof: Let $\pi_1$ be a path which performs the Follow motion on $C$, as shown in Fig. 7. According to Proposition 5, the covered cells on $C$ are on the left side of $C$. Let $C_{sub}$ be the covered subsegment of $C$. After $\pi_1$, let $c_1, c_2, ..., c_p$ be the uncovered cells on $C$ from right to left. $c_{p+1}$ is the rightmost covered cell on $C$ by $\pi_1$. Let $C'$ be any contour in $C$ and $c'_1, c'_2, ..., c'_q$ be the uncovered cells on $C'$ from right to left. Then $c'_{q+1}$ is the rightmost covered cell on $C'$. By Proposition 3, we know $c'_k$ is accessible from $c_k$. If $q > p$, then $c'_{q+1}$ is accessible from a cell which is left to $c_{p+1}$. Then $\pi_1$ should stop the Follow motion before reaching $c_{p+1}$ and continue to Advance to the further contours. So $q \leq p$. That is, any contour in $C$ (if exists) has uncovered cells no more than $C$ has.

With Proposition 6, we can prove the following lemma.

Lemma 1: If a path performs the Follow motion on a contour $C$, $C$ cannot be a saturated contour.

Proof: Let $\pi_1$ be a path which performs the Follow motion on $C$. By Proposition 6 we know the contours in $C$ has uncovered cells no more than $C$ has. Meanwhile, the following paths’ Advance motions will newly cover at least one cell on each contour in $C$. So when $C$ is fully covered, the contours in $C$ will also be fully covered. So the remaining paths will not visit $C$ anymore. Thus $C$ cannot be saturated.
Since $B_j \leq B_j^*$, $n_j \leq n_j^*$, we get $B_{j-1} \leq B_{j-1}^*$. By Lemma 2, we know
\[ n_{j-1} \leq \frac{B_{j-1}}{B/2 - d(C_{j-1})}, \]
\[ n_{j-1}^* \geq \frac{B_{j-1}^*}{B/2 - d(C_{j-1})}. \]
Since $B_{j-1} \leq B_{j-1}^*$, we get $n_j \leq n_j^*$. So the statement is also true for $k=j-1$. Then we can conclude that Lemma 3 is true for all the saturated contours.

**Theorem 1:** Let $m$ be the number of paths from Algorithm 1 and $n^*$ be the number of paths in $OPT$. Then $m \leq 4n^*$.

**Proof:** From Lemma 3, it is easy to see that $|\Pi_{sol}| \leq 2n^*$. By Proposition 4, one path in $\Pi_{sol}$ actually consists of two paths directly from Algorithm 1. So $m \leq 4n^*$.

**Theorem 2:** Let $l$ be the total length of paths from Algorithm 1, and $l^*$ be the total length of $OPT$. Then $l \leq 8l^*$.

**Proof:** There are $m$ paths from Algorithm 1. So $l \leq mB$. $OPT$ has $n^*$ paths. No two paths can have a total length less than $B$. Otherwise they can be combined to form a shorter path. So $l^* \geq \frac{B}{2}n^*$. Since $m \leq 4n^*$, $l \leq 8l^*$.

**VI. COVERAGE STRATEGY FOR GENERAL ENVIRONMENTS**

The main idea for covering non-contour-connected environments is to partition them into contour-connected parts and run Algorithm 1 on each of them. We first discuss where Lemma 1 fails for non-contour-connected environments. Then we show how to partition the environments and prove that the performance can be bounded by the number of reflex vertices of the environments.

Fig. 8 shows a simple non-contour-connected area. From $u$, the contour $C$ breaks into two segments $E_1$ and $E_2$. $C$ forks into two branches, $E_1$ and $E_2$. Consider the case when $E_1$ is small and needs less than $|E_1|$ paths, but $E_2$ is large and needs more than $|E_2|$ paths. In this case, the paths will perform the Follow motion before entering $E_2$, but the contours in $E_2$ can still be saturated. This violates Lemma 1.

Now we explain how to partition the environment into contour-connected subareas. Assume that $S$ is at an acute vertex (If not, we just split the environment into at most four parts by the horizontal and vertical lines which cross $S$). We use a line of the same direction as the first contour to sweep the environment from $S$. When the line sweeps to a reflex vertex $v$ on the contour $C$, there are four possible cases, as shown in Fig. 9. (i) $C$ is a straight line, and $v$ is at an endpoint of $C$, as shown in Fig. 9(a). In this case, a reflex vertex does not affect the sweeping process. (ii) The contour $C$ at $v$ is a straight line and from $v$, $C$ breaks into two segments, as shown in Fig. 9(b). We form one contour-connected subarea by the previously swept area. Then we start two new sweeping processes from the two segments of $C$ split by $v$. In this case, the reflex vertex forms one subarea and starts two new sweeping processes. (iii) The next contours consist of two line segments, as shown in Fig. 9(c). We call the position where the two line segments meet as turning cells. We use these turning cells as a virtual boundary of the environment. We continue the sweeping process at one side of this virtual boundary, and start a new sweeping process from this reflex vertex on the other side of this virtual boundary. In this case, the reflex vertex starts one more sweeping process. (iv) Two sweeping processes meet at $v$ (which implies that $v$ is a vertex of an inner obstacle), as shown in Fig. 9(d). We stop the two sweeping processes to form two subareas. Then we start one new sweeping process from $C$. In this case, the reflex vertex stops two sweeping processes and starts a new one. When there are no more contours in the current sweeping process, the area swept by this process forms a new subarea. Fig. 10 shows a partition example.

Let $r$ be the number of reflex vertices of the environment. We run Algorithm 1 on each subarea to get our solution.

**Lemma 4:** The sweeping processes generate at most $2r + 4$ contour-connected subareas of the environment.

**Proof:** Each reflex vertex starts at most two more sweeping processes. If $S$ is not at an acute corner, there
are at most four sweeping processes from $S$. So the number of subareas are at most $2r+4$.

Theorem 3: Let $n$ and $l$ be the number of paths and the total path length to cover the given environment by our algorithm. Let $n^*$ and $l^*$ be the number of paths and total path length to cover the same environment by the optimal solution. $n \leq 4(2r+4)n^*$, $l \leq 8(2r+4)l^*$.

Proof: Let $n^*_i$ and $l^*_i$ be the number of paths and total path length when $OPT$ covers the $i$-th subarea from the sweeping process. It is easy to see that $n^*_i \leq n^*$, $l^*_i \leq l^*$. Let $n_i$ and $l_i$ be the number of paths and total path length when Algorithm 1 covers this subarea. According to Theorem 2, $n_i \leq 4n^*_i$, $l_i \leq 8l^*_i$. There are at most $2r+4$ subareas from the sweeping process. Thus

$$n = \sum_{i=1}^{2r+4} n_i = \sum_{i=1}^{2r+4} 4n^*_i \leq \sum_{i=1}^{2r+4} 4n^* = 4(2r+4)n^*,$$

$$l = \sum_{i=1}^{2r+4} l_i = \sum_{i=1}^{2r+4} 8l^*_i \leq \sum_{i=1}^{2r+4} 8l^* = 8(2r+4)l^*.$$

VII. FIELD EXPERIMENT

The goal of the experiment is to validate the uniform energy consumption model, and to demonstrate the paths from Algorithm 1 in an environmental monitoring application.

We use a DJI Phantom 3 SE UAV to execute the paths. One reason to use the UAV is that it can fly in four directions (front, back, left, right) without rotation, which allows us to ignore the energy consumed by rotation. We select an open field in Cedar Creek Ecosystem Science Reserve, MN, USA to do the experiment, which is shown in Fig. 11. Most parts of this area are not easily accessible to humans, which makes it ideal for a UAV to cover it. We use the App Litchi to guide the UAV to follow the paths. We also read the battery’s remaining energy from Litchi and use that to measure the energy consumption.

To validate the uniform energy consumption model, we design the flight plan, as shown in Fig. 12 and let the UAV follow it. We record the remaining energy of the battery along the way. We did two flights with the same battery. The straight lines in Fig. 13 illustrate that the uniform energy consumption model is reasonable.

Next, we plan the paths to cover the area. For safety we want the battery to still have enough energy after following each path. We set the energy budget as 1200 meters for each path. The flight altitude is set to be 10 meters, and the UAV’s camera is able to cover a $20m \times 20m$ cell. Algorithm 1 returns five paths. The actual flight trajectory is shown in Fig. 14. If we count the cells visited by Retreat motion, the fifth path is unnecessary. The total area is 46400m$^2$. One path is able to cover an area of $1200 \times 20 = 24000m^2$ at most. So $OPT$ needs at least 2 paths to completely cover this area. Our solution has five paths, less than four times of $OPT$, as given by Theorem 1.

VIII. CONCLUSION AND FUTURE WORK

In this paper, we presented an algorithm for the energy-constrained coverage path planning problem. For contour-connected environments, the approximation ratio of the algorithm is shown to be 4 for minimizing the path number and 8 for total length. For arbitrary environments, the approximation ratios are $4(2r+4)$ and $8(2r+4)$ respectively, where $r$ is the number of reflex vertices of the environment.

Fig. 11. The Experiment filed.

(a) path 1  (b) path 2  (c) path 3  (d) path 4  (e) path 5  
(f) whole trajectory

Fig. 12. The flight plan to test battery consumption.

Fig. 13. The battery’s remaining energy in terms of distance travelled.

Fig. 14. The actual flight trajectory of the UAV. When plotted together in (f), previous paths have higher priority.
In the present work, we restrict the robot motion to be rectilinear. One direction for future research is to extend the algorithm for unconstrained motion. Designing a constant-approximation algorithm for arbitrary problems is another important direction.

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[3] Y. Choi, T. Lee, S. Baek, and S. Oh. Online complete coverage consisting of multiple segments. Let $C_{next}$ be the contour that is one unit further to $S$ than $C$. If $c$ is not adjacent to the boundary, then $C_{next}$ must have the three cells that are $c$’s neighbors (colored green). So $C_{next}$ is also a polyline consisting of at least two segments and has a turning vertex. Let $c'$ be the furthest turning vertex in $C$. Then $c'$ must be adjacent to the boundary. Otherwise, there will be a further turning vertex than $c'$. So $c'$ is a split cell, and the environment is not contour-connected. Thus, a contour in contour-connected environments must be a line segment.

Proposition 2: Let $C$ and $C'$ be two adjacent equi-distance contours. Then $||C|| - ||C'|| \leq 1$.

Proof: Let $C$ be a contour with $n$ cells. According to Proposition 1, $C$ is a line segment. Let $C'$ be a contour adjacent to $C$ (closer or further to $S$). The cells $C'$ must be adjacent to cells on $C$, since a cell on $C'$ must have a unit distance to a cell on $C$. So there are $(2n + 2)$ possible locations for cells on $C'$, as shown in Fig. 15(b). But $C'$ is a line segment, it cannot have $(n + 2)$ cells on it. Otherwise, it will have a turning vertex. Neither $C'$ can have less than $(n - 1)$ cells, since in that case one cell on $C$ will be adjacent to the boundary and be a split cell, which contradicts the contour-connected assumption. So $||C|| - ||C'|| \leq 1$.

Proposition 3: Let $C$ and $C'$ be two contours in a contour-connected environment and $C''$ is further than $C$, $c_1, c_2, ..., c_p$ are the cells on $C$ from one side to the other. $c_1', c_2', ..., c_q'$ are the cells on $C'$ from the same side to the other. $c_i$ is defined to be $c_p$ if $i > p$. $c_i'$ is defined to be $c_q'$ if $i > q$. Then $c_i'$ is accessible from $c_i$ for all $i$.

Proof: We first prove that this proposition is true for two adjacent contours. Let $C'''$ be a contour with $d(C''') = d(C') + 1$. Let $c_{1}', c_{2}', ..., c_{15}'$ be the cells on $C'''$ starting from the same side as $c_1$. We prove this by induction.

$C'''$ has two candidate positions, $c_{15}'$, and $c_{15}'$, as shown in Fig. 16. Both of them are accessible from $c_1$. Suppose when $k = j$, $c_j'$ is accessible from $c_j$. Then there are two possible locations, $c_{15}'$ and $c_{15}'$, for $c_j'$. They again lead to two possible locations, $c_{(j+1)}$ and $c_{(j+1)}$, for $c_{j+1}'$. We can see that both $c_{(j+1)}$ and $c_{(j+1)}$ are accessible from $c_{j+1}$. By induction, $c_j'$ is accessible from $c_i$ for all $i$. The proposition is true for two adjacent contours.

For any two contours $C$ and $C'$ with $d(C') > d(C)$, if all the contours between $C$ and $C'$ have more or equal number of cells, it is easy to see that $c_i'$ is accessible from $c_i$. Suppose an intermediate contour $C_k$ has $k$ cells, where $k \leq i$, as
shown in Fig. 17. We know $c_{kk}$ is accessible from $c_i$. We can also know that $c'_i$ is accessible from $c_{kk}$. Thus $c'_i$ is accessible from $c_i$. The proposition holds for any two contours. ■